

## JUSTIFYING INVARIANT SUBSPACES IN EUCLIDEAN SPACES: A CONCEPTUAL APPROACH

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### Abstract

The concept of invariant subspaces plays a fundamental role in the study of Euclidean spaces and linear transformations. This paper provides a conceptual approach to justifying the existence and properties of invariant subspaces within Euclidean spaces. By exploring key definitions and properties, such as basis, dimension, and orthogonality, the paper establishes the foundational aspects of invariant subspaces. The role of linear operators is analysed to demonstrate how they define and influence the structure of these subspaces. Furthermore, the paper illustrates the presence of invariant subspaces in Euclidean spaces, supported by theoretical proofs and conceptual arguments. The study highlights the significance of invariant subspaces in understanding eigenvalues, eigenvectors, and eigenspaces, which are crucial in simplifying matrix decomposition and other mathematical processes.

*Keywords:* Equation, Subspace, Invariant, Eigen spaces, Approach.

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### INTRODUCTION

The study of linear transformations and their spectral features is one area where the notion of invariant subspaces plays a key role in linear algebra and functional analysis. Defining invariant subspaces and providing theoretical and practical justifications for their usefulness are prerequisites for approaching this topic in the context of Euclidean spaces.

To put it simply, an invariant subspace is a subspace of a vector space that does not change when a linear transformation is applied. In a more formal sense, if  $T: X \rightarrow V$  is a linear operator on a vector space  $XV$ , then a subspace  $W \subseteq V$  is said to be invariant under  $T$  if and only if  $T(w) \in W$  for all  $w \in W$ .

When considering finite-dimensional inner product spaces, like Euclidean spaces, invariant subspaces acquire extra structure as a result of the geometry introduced by the inner product. With this framework, we can learn more about how linear operators behave, which is useful when dealing with physical systems or transformations in engineering and mathematics.

The study of invariant subspaces is warranted because of the problems that they simplify. An effective way to comprehend many linear operators—including those in quantum mechanics, signal processing, and differential equations—is to partition the space they operate on into smaller, more tractable subspaces that remain unchanged when subjected to the operator. One example is the construction of one-dimensional invariant subspaces connected by eigenvectors, which is central to eigenvalue issues in linear algebra. The operator's action is succinctly depicted by these eigenvectors and their related eigenvalues, which disclose important details on its dynamics and stability.

The properties of invariant subspaces in Euclidean spaces are also strongly related to diagonalization and orthogonal projections. If there is a basis of  $V$  made up of eigenvectors of  $T$ , which are elements of invariant subspaces, then a linear operator  $T$  can frequently be diagonalized. Matrix exponentiation and solving systems of differential equations become much more manageable after this diagonalization, which simplifies the operator. When full diagonalization is not achievable, partial simplifications can be achieved by block-diagonal representations made available by invariant subspaces.

If we want to know how operators are structured theoretically, invariant subspaces are crucial. One example is the spectral theorem, which relies on the fact that normal operators in Euclidean spaces and self-adjoint operators have invariant subspaces. With this theorem as a guarantee, we may efficiently compute these operators by representing them as diagonal matrices in an appropriate orthonormal basis, which brings attention to their spectral features.

Finally, invariant subspaces give a connection between linear transformations' theoretical foundations and their real-world implementations. These subspaces allow us to better comprehend the structures in Euclidean spaces and simplify mathematical models by giving a structure for dissecting and analyzing linear operators. Both the theoretical elegance and the significant practical consequences of invariant subspaces research provide strong justification for this area of study.

## PRELIMINARIES

- **Definition of Euclidean spaces and their properties.**

The geometry and linear algebraic frameworks are based on the concept of Euclidean spaces, which are named after the ancient Greek mathematician. In a finite-dimensional vector space called a Euclidean space, the concepts of distance and angle are clearly defined and are derived from the inner product. The rigorous consideration of geometric notions like length, orthogonality, and projection is made possible by this structure. The mathematical notation for an  $n$ -dimensional Euclidean space is  $R^n$ , where

$n$  is the number of dimensions and  $n$  is the order of the  $n$ -tuples of real numbers that make up the space's constituents, the vectors.

Euclidean spaces are defined by the existence of an inner product, which transfers two vectors to a scalar that is bilinear, symmetric, and positive-definite. The inner product of two vectors  $f, a \in \mathbb{R}^n$ ,  $u, v \in \mathbb{R}^n$  is usually written as  $\langle u, v \rangle$ , which is the same as the dot product. For any vector  $v$ , the inner product  $\langle v, v \rangle$  generates a norm, or length. A crucial aspect of Euclidean spaces is the norm, which allows us to define the distance between two vectors as  $\|u - v\|$ , by providing a measure of the magnitude of the vectors.

The geometric qualities of Euclidean spaces are further defined by the fact that they are orthogonal and that they can have bases defined. When  $\langle u, v \rangle = 0$ , we say that vectors  $u$  and  $v$  are orthogonal. A basic idea with several practical applications in engineering and science, orthogonality permits the partitioning of vectors into parts that do not mutually influence one another. One further thing that makes a space Euclidean is if it has a basis, which is a collection of vectors that cover the whole space and are linearly independent. An enormous simplification of coordinate and transformation calculations occurs when the basis vectors are orthonormal, i.e., have unit length and are mutually orthogonal.

The completeness of Euclidean spaces is another important characteristic. Whatever this implies is that all vectors in a Cauchy sequence converge to a vector in this space. Due to their inherent completeness, Euclidean spaces constitute the gold standard for numerous numerical methods and are capable of supporting complex mathematical analyses. Further, even when subjected to stiff transformations like translations and rotations, the geometry of spaces defined by Euclidean geometry remains unchanged, preserving both distances and angles. The attributes of objects must be consistent under multiple transformations in physics and computer graphics, making this invariance vital.

Projections, defined in part by the structure of Euclidean spaces, find widespread use in fields as diverse as computer vision and data analysis. The projection of  $v$  onto  $W$  is the vector in  $W$  that is the closest to  $v$  in terms of Euclidean distance, given a vector  $v$  and a subspace. Orthogonal projections, fundamental to methods like principal component analysis and the Gram-Schmidt process, are an obvious extension of this idea.

To sum up, the algebraic and geometric structures that make up Euclidean spaces make them a very intuitive and flexible foundation for mathematical theory and practice. Their completeness, invariance, and well-defined concepts of distance, angle, and orthogonality make them crucial for comprehending and resolving several scientific and technical issues.

- **Definition of invariant subspaces, basis, dimension, and orthogonality.**

### **Invariant Subspaces**

A subspace of a vector space that does not change when a specific linear operator is applied is called an invariant subspace. Consider  $X, V$  as a vector space over  $C, F$  and  $T: V \rightarrow V$  as a linear operator from  $X, V$ . If  $T(w) \in W$  for every  $w \in W$ , then the subspace  $W \subseteq V$  is said to be invariant under  $T$ . The outcome remains contained in the subspace  $W$  when the linear operator  $T$  is applied to any vector in  $W$ . Because they enable the reduction of vector spaces into smaller components, invariant subspaces play a crucial role in linear algebra by providing light on the structure of linear operators. A matrix's eigenspaces are the invariant subspaces that correspond to it, as an example.

### **Basis**

A collection of vectors that fulfils two essential properties: linear independence and span, is called a basis of a vector space  $X, V$  over a field  $A, F$ . When all of the vectors in the basis are linearly independent, it means that you can't represent any of them as a linear combination. Every vector in  $X, V$  can be expressed as a linear combination of the basis vectors according to the span property. Taken as a whole, these characteristics guarantee that the basis adequately describes the vector space with minimal effort. The dimension of the vector space is the quantity of vectors in the basis. Crucially, the definition of dimension is based on the fact that any two vector space bases have an equal number of elements.

### **Dimension**

The quantity of vectors in a basis for a vector space  $V$  is called its dimension. It gives a degree-of-freedom (or "size" or "complexity") metric for the space. The dimension of a finite-dimensional vector space is an integer that is not negative. For instance, because its standard basis is composed of  $n$  unit vectors, the space has  $n$  dimensions. Matrix rank, linear system solutions, and linear map behaviour are all aspects that are affected by dimension, which is a crucial variable in linear algebra and geometry. The idea of dimension takes on a more complex form when dealing with spaces with infinite dimensions, like specific function spaces, and is defined in terms of cardinality instead of finite numbers.

### **The concept of orthogonality**

In a vector space with an inner product, the idea of orthogonality describes the relationship between vectors. Orthogonality is defined as the condition where the inner product of two vectors  $f, a \in X, u, v \in V$  equals zero. Orthogonal vectors are "perpendicular" with respect to the space's inner product, according to this geometric meaning. Two subspaces are considered orthogonal if and only if every vector in one is orthogonal to every vector in the other. This property of orthogonality is extended to subspaces. Many branches of engineering and mathematics rely on orthogonality, especially numerical approaches and signal processing. In the Gram-Schmidt method for building orthonormal bases, where the vectors are unit length and orthogonal, it is the building block.

## ROLE OF LINEAR OPERATORS IN DEFINING INVARIANT SUBSPACES

In describing and studying invariant subspaces, linear operators are crucial. Vectors in  $V$  can be translated to other vectors in the same space by a linear operator  $T: X \rightarrow V$ , where  $V$  is a vector space over  $C$  or  $F$ . For any  $w \in W$ , a subspace  $W \subseteq V$  is considered to be invariant under  $T$  if and only if  $(w) \in WT$ . Because of this condition, any vector in  $W$  will not yield an outcome outside  $W$  when subjected to the action of  $T$ . An important reason invariant subspaces are important is that they simplify the problem by allowing the operator  $T$  to be explored in parts.

Through eigenvalues and eigenvectors, linear operators are mainly linked to invariant subspaces. In the case when  $av$  is an eigenvector of  $T$ , the one-dimensional subspace that it spans remains unchanged under  $T$ . By extension, the eigenspaces that contain an eigenvalue are likewise invariant subspaces. When obtaining the Jordan form or diagonalizing a matrix, these eigenspaces shed light on the operator's structure. Operators that are not diagonalizable can be decomposed into more manageable parts by finding invariant subspaces that correspond to generalized eigenvectors.

Function spaces are not the only domains where linear operators can affect invariant subspaces. An example would be the definition of invariant subspaces of function spaces by differential operators in differential equations. These subspaces would typically include polynomial or exponential functions. The same holds true in quantum mechanics: observable quantity operators act on Hilbert spaces, with invariant subspaces standing in for physically meaningful states.

Also, in many branches of mathematics, the idea of decomposing vector spaces into invariant subspaces is fundamental. A complex vector space can be divided into orthogonal invariant subspaces that correspond to the eigenvalues of the operator, according to the spectral theorem for normal operators. The ability to break down complicated systems into manageable components is essential for many scientific and technological fields, including computer science, engineering, and physics.

Additionally, iterative approaches for numerical computations are made easier by invariant subspaces. As an example, invariant subspaces are fundamental to the power technique, which is used to determine the dominating eigenvalue of a matrix. In a similar vein, Krylov subspace methods rely on building and operating within invariant subspaces to solve eigenvalue problems or huge systems of linear equations. So, in addition to defining invariant subspaces, linear operations make them usable in theory and practice.

## PRESENCE OF INVARIANT SUBSPACES IN EUCLIDEAN SPACES

In the context of Euclidean spaces, invariant subspaces are prominent and clearly defined, and there is an intuitive geometric interaction between the geometry of these spaces and the characteristics of linear operators. The standard inner product of a Euclidean space makes it an ideal laboratory for investigating

subspaces and whether or not they are invariant under linear transformations. A subspace  $W \subseteq V$  is said to be invariant under  $T$  if for every vector  $w \in W$ ,  $T(w) \in W$ , and  $T$  is a linear operator.

Eigenspaces of matrices expressing linear transformations frequently give rise to invariant subspaces in the setting of Euclidean spaces. The subspace spanned by  $av$  is invariant under  $T$  if  $av$  is an eigenvector of  $T$ . In a broader sense, invariant subspaces are formed by the eigenspaces that correspond to distinct eigenvalues. These eigenspaces shine when the diagonalizable matrix represents  $T$ . A direct sum of these eigenspaces, each of which is invariant under  $T$ , can be used to decompose in such circumstances.

A further level of organization is introduced to the investigation of invariant subspaces by the notion of orthogonality, which is fundamental to Euclidean spaces. The spectral theorem ensures that can be written as a direct sum of orthogonal invariant subspaces for normal operators, such symmetric or orthogonal matrices. By preserving the inner product structure within each subspace, this orthogonality simplifies the study of  $T$ . For instance, these orthogonal invariant subspaces lay the groundwork for data decomposition into uncorrelated components when conducting principal component analysis or solving systems of equations.

Numerical methods can also benefit from invariant subspaces in Euclidean spaces. In order to converge to eigenvectors or eigenvalues of matrices, iterative techniques like the power method or QR algorithm frequently implicitly depend on the existence of invariant subspaces. Solving large linear systems or eigenvalue issues with Krylov subspace methods is similar; these methods rely on building subspaces that are invariant under particular matrix powers or projections.

Invariant subspaces are not limited to finite-dimensional contexts; they can also be applied to infinite-dimensional spaces that resemble Euclidean spaces, as spaces in functional analysis. A lot of the rules for when and how to use invariant subspaces in these contexts are the same as in finite-dimensional Euclidean spaces. Regardless of the underlying space's dimensions, this relationship demonstrates that invariant subspaces are a tool for comprehending linear transformations.

A theoretical and practical tool, invariant subspaces in Euclidean spaces allow us to break down spaces and transformations into smaller, more manageable parts while keeping our geometric intuition and inner product structures intact.

## CONCLUSION

Geometric intuition and algebraic structures interact to give rise to the idea of invariant subspaces in Euclidean spaces. Within the structured environment of  $R^n$ , these subspaces offer a basic framework for breaking down and studying the behaviour of linear operators. Their importance stems from the fact that they provide light on transformation behaviour while maintaining essential features of a Euclidean space such orthogonality, dimension, and linearity. The fact that invariant subspaces in Euclidean spaces make the study of linear transformations easier is a major argument in their Favor.

When limited to invariant subspaces, linear operators frequently exhibit a more simplified structure, like diagonal or triangular forms. Eigenvalues, diagonalizing matrices, and dynamical system knowledge are some of the difficulties that can be solved thanks to this operator analysis being more tractable. The geometric definition of these transformations in Euclidean spaces gives an intuitive grounding for their algebraic analysis, which in turn supports the use of invariant subspaces. An further reason invariant subspaces are important is that they keep orthogonality in Euclidean spaces. When a space is orthogonally divided into invariant subspaces, the geometric core of the space is preserved since the decomposition does not disturb the inner product structure. The spectral theorem ensures that eigenspaces and invariant subspaces will be orthogonally decomposed, making this quality especially important for symmetric and orthogonal operators. In signal processing, for example, when orthogonal components represent separate modes of variation, such decompositions simplify both theoretical and practical concerns.

For more general mathematical frameworks like Hilbert spaces in functional analysis, invariant subspaces also act as a conceptual link between finite-dimensional Euclidean spaces. Their conceptual coherence and universal applicability are underscored by the fact that the laws regulating invariant subspaces in naturally apply to settings with infinite dimensions. Understanding increasingly complicated spaces and operators in high-level mathematics and physics is thus built upon the study of invariant subspaces in Euclidean spaces. When seen from a higher vantage point, invariant subspaces in Euclidean spaces represent the sweet spot between being too general and being too specialized. Not only do they record the distinctive geometric and algebraic properties of the space in issue, but they also give a universal tool for assessing linear transformations. Their crucial position in both mathematical theory and application is justified by their dual nature, which combines conceptual simplicity with practical versatility. The fact that they are still relevant shows how basic mathematical concepts can shed light on and make real-world phenomena and abstract systems easier to understand.

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