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ARITHMETIC PROGRESSION AND BINARY RECURRENCE

Pritee[1] and Jakhar, Manjeet Singh[2]

[1]Research Scholar, NIILM University, Kaithal, Haryana

[2]Associate Professor, Department of Mathematics, NIILM University, Kaithal, Haryana

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Abstract

When it comes to Galois theory, the idea of a field extension is considered to be one of the most fundamental concepts. In Galois theory, the field that is now being explored is used as an input, and the features of extension fields are investigated in relation to the field that is currently being investigated. It focuses on something called "Galois extensions," which are simply fields that possess specific symmetry features. These fields are referred to as "Galois extensions." Because of these features, we are now in a position to identify a relationship between the structure of the Galois group that is associated with the field extension and the structure of the field extension itself. This connection is between the structure of the Galois group and the structure of the field extension. A Galois group is a group that is produced when the auto orphisms of an extension field that fix the element wise structure of the base field provide the group. This results in the formation of the Galois group. A Galois group is a group that is built by the auto orphisms of the extension's elements. An extension's Galois group is a group that is constructed by these auto orphisms. These auto orphisms may be thought of as permutations of the roots of the polynomial equation that defines the extension in its most basic form.

Keywords : Terms, Arithmetic, Progression

Introduction

Arithmetic Progression

One of the most well-known theorems that Sylvester has developed is , which asserts that a product of k successive terms, each of which is more than k, may be divided by a prime number that is greater than k. This theorem states that the product of k successive terms is equal to the sum of those terms. In the event that n is bigger than k and we had two positive integers, n and k, for example, we may assert that

$$P(n(n+1)\cdots(n+k-1)) > k. \quad \ldots\ldots\ldots\ldots\ldots(1)$$

As a consequence of the fact that the claim is not true for each and every n, it is necessary to take into account that this outcome is something that is not straightforward. The claim is not valid when n = 1 for each and every k, hence it is required to make the assumption that n is bigger than k given that this is the case. Make it possible for the numbers n, d, and k to have a value that is positive. Let us now make the assumption that if we are given n, d, and k, then the value of gcd (n, d) is equal to 1. After this point, this will continue to be the case. It was demonstrated by Sylvester that for d values higher than one.

$$P(n(n+1)\cdots(n+k-1)) > k \quad \text{if} \quad n+d \geq k > 2. \quad \ldots\ldots\ldots\ldots\ldots(2)$$

In their study, Shorey and Tijdeman expanded upon the finding that Sylvester had made regarding the value of d that was higher than just one. That was established by them.

$$P(n(n+d)\ldots(n+(k-1)d) > k \text{ for } k \geq 3 \text{ unless } (n,d,k) = (2,7,3). \quad \ldots\ldots\ldots\ldots(3)$$

It is of utmost importance to take note of the fact that the numbers n = 1, d = 2, r − 1, and r > 1 provide an unlimited number of counterexamples for k = 2, hence showing that k is higher than or equal. The conclusion that can be drawn from equations (3.1) and (3.2) is that, given positive values of n, d, and k, where n is greater than k if d is equal to one and k is greater than three if d is greater than one, there exists a term n + id with 0 ≤ i < k that can be divided by a prime that is greater than k. However, the equation (n, d, k) = (2, 7, 3) is an exception to this rule.

OBJECTIVE OF THE STUDY

1. To study Terms of An Arithmetic Progression
2. To study Galois Group Of Some Orthogonal Polynomials

A similar inquiry is investigated here:

When we given a set of positive numbers n, d, and k, is there a value of i that meets the conditions that P(n + id) is more than k, 0 is less than i, and n + id is not an odd number? If so, then what is the value of i that satisfies these constraints?

Bravo, Das, Guzman, and Laishram responded to this inquiry by presenting the conclusion that they arrived to over the course of their study. This was in response to the situation in which d is more than one and k is greater than or equal to six.

It is determined that is the conclusion. Let n be more than one, d be greater than one, and k be greater than six. This is based on the assumption that the value of gcd(n, d) is equal to one. As a result of the fact that n + id is an odd integer, the chance that P(n + id) is greater than k is higher than k in this particular scenario. Furthermore, there is a minimum of one i, where 0 is either less than or equal to k depending on the situation.

First, the result that was presented earlier, which was a generalised Sylvester 3.1 result, is taken into consideration, and then it is added to. Furthermore, not only does it have significance in and of itself, but it is also essential for the demonstration of our hypotheses.

Three-thirds theorem As a reference, Let n be more than one, d be greater than one, and k be greater than six. This is based on the assumption that the value of gcd(n, d) is equal to one. As a result of the fact that n + id is an odd integer, the chance that P(n + id) is greater than k is higher than k in this particular scenario. Furthermore, there is a minimum of one i, where 0 is either less than or equal to k depending on the situation.

Therefore, in order to demonstrate that Theorem is correct, we need to arrive at the following conclusion, which is an improvement on the one that Laishram and Shorey came up with. This is because of the fact that the previous conclusion was flawed.

Lemma. Using the equation gcd(n, d) = 1, let n be more than or equal to 1, d be greater than or equal to 2, and k be greater than or equal to three. This will serve as the basis for the equation. We are able to determine the value of n by using this equation. When that time came,

$$P\left(\prod_{i=0}^{k-1}(n+id)\right) > 2k+1 \quad \text{.............(4)}$$

unless k = 3, (n, d) ∈ {(1, 124), (7, 118)}.

Proof of Lemma:

Assume d > 2 and n ≥ 1. The variable d should be an even number that is not a power of two, and the variable k should be more than four. Allow gcd(n, d) to be 1 in this case. It should come as no surprise that is the right answer when (n, d, k) = (1, 4, 7). Because of this, we have made the decision to go with.

$$P\left(\prod_{i=0}^{k-1}(n+id)\right) \leq 2k+1. \quad \text{..............(5)}$$

Let k = 3. Then

$$P(n(n+d)(n+2d)) \leq 7. \quad \text{..............(6)}$$

Given that d is even and n is odd, and with 0 ≤ i ≤ j ≤ 2, the generalised cumulative distribution of n + id, n + jd is nothing more than one. One of the outcomes of the equation {n, n + d, n + 2d} is the expression {3 a, 5 b, 7 c}, when n is larger than 1. When n has reached the value of one, then

$$(n+d)(n+2d) = (1+d)(1+2d) = 3^a 5^b 7^c. \quad \text{..............(7)}$$

It is possible to get the equation (1) − (9) that is stated in Lemma by employing the formula n + (n + 2d) = 2(n + d). This equation may be obtained by utilizing the formula. From the answers that are supplied in Lemma, we are able to draw the conclusion that d is equal to two, and that n is a member of the set of numbers or that (n, d) is a member of the set of numbers. In light of the fact that d > 2 is not a power of 2 and 3 - d, the assertion of the lemma asserts that the set has a subset that is comprised of the elements (n, d).

Galois Group Of Some Orthogonal Polynomials

For a positive real number α, let $L_m^{(\alpha)}(x)$ Please indicate the Laguerre polynomial of degree m, which is given by

$$L_m^{(\alpha)}(x) = \sum_{j=0}^{m} \frac{(m+\alpha)(m-1+\alpha)\cdots(j+1+\alpha)}{(m-j)!j!}(-x)^j. \quad \text{……………(8)}$$

We set

$$v^m m! L_m^{(u/v)}\left(-\frac{x}{v}\right) := g(x) = g(x, u, v)$$

$$= \sum_{j=0}^{m} \binom{m}{j} (vm+u)\cdots(v(j+1)+u)x^j, \quad \text{…………………………(9)}$$

in which u and v are both greater than zero and gcd(u, v) equals 1. In the event that the Galois group of an irreducible polynomial f of degree m contains Am, then it is either Am or Sm, depending on whether or not the discriminate of f is a square. This is a well-known fact. Within the context of g(x), Schur demonstrated that the discriminate may be understood as

$$D_m^{(u,v)} = \prod_{j=2}^{m} j^j \left(\frac{u}{v} + j\right)^{j-1}. \quad \text{……………(10)}$$

We write

$$D_m^{(u,v)} = bY^2, \quad Y \in \mathbb{Q}, \quad \text{……………(11)}$$

where

$$b = \begin{cases} \dfrac{3\cdot 5\cdots m\cdot (u+2v)\cdots (u+(m-1)v)}{v^{\delta_1}} & \text{if } m \equiv 1,3 \pmod 4, \\ \dfrac{3\cdot 5\cdots (m-1)\cdot (u+2v)\cdots (u+mv)}{v^{\delta_2}} & \text{if } m \equiv 0,2 \pmod 4, \end{cases}$$ ……………(12)

with

$$\delta_1 = \begin{cases} 0 & \text{if } m \equiv 1 \pmod 4, \\ 1 & \text{if } m \equiv 3 \pmod 4, \end{cases} \quad \text{and} \quad \delta_2 = \begin{cases} 0 & \text{if } m \equiv 0 \pmod 4, \\ 1 & \text{if } m \equiv 2 \pmod 4. \end{cases}$$

……………(13)

For any irreducible g(x), it is sufficient to determine when b, whose numerator is a product of two blocks of integers in arithmetic progression, is a square. This process is known as the sureness of the numerator. Let's say that (u, v) equals (1, 1). Schur demonstrated that the function g(x) is irreducible. A further observation that he made was that if m is odd, or if m is even, and m plus one is a square, then b is a square. Therefore, the Galois group of g(x) is Am in these particular instances. In the case where m is even and m plus one does not constitute a square, then the Galois group is Sm. In the general case $L_m^{(u/v)}$, This was demonstrated by Filaseta and Lam, who demonstrated that the polynomial g(x) is irreducible over rationals for all positive integers m with the exception of a finite number of them. $\frac{u}{v}$ is an integer that is negative. A complement to the finding that Filaseta and Lam obtained was offered by Hajir in the form of a computation of the Galois group of g(x) when m is a high number. The explicit values m0 that Akhtari and Saradha provided in depended on u and v in such a way that for all possibilities, $m \geq m_0, g(x)$. can never be reduced. In addition, they demonstrated that there is a maximum of n0 the number of polynomials g(x) that might not be irreducible, which makes the conclusion of explicit for small values of v. Please go to for results on polynomials that are more broad. Through the use of the formula for b that is presented, we are able to make the result of Hajir explicit for specific values of (u, v) in this composition. The example where v is equal to one and u is between one and ten was handled by Banerjee, Filaseta, Finch, and Leidy. By demonstrating the following theorem, we.

Binary Recurrence

If n is a natural integer, then it becomes a balanced number. $1+2+\cdots+(n-1) = (n+1)+\cdots+(n+r)$ in the case of a certain r that is natural. Panda and Ray determined that a natural number n might be considered a co balancing number with a co balancer r if they made a little adjustment to the equation that defines balancing numbers. $1+2+\cdots+n = (n+1)+\cdots+(n+r).$ The concepts of co balancing and balancing numbers have been advanced in a variety of different ways throughout the years. As an alternative to the natural numbers that are often used in the definitions of balancing and co balancing numbers, Panda suggested that an arbitrary integer sequence might be utilized instead. $a_m, m = 1, 2, \cdots.$ He referred to one of these terms as the sequence balancing number if $a_1 + a_2 + \cdots + a_{n-1} = a_{n+1} + \cdots + a_{n+r}$ With regard to a particular r, he made reference to a series co balancing number; nevertheless, if $a_1 + a_2 + \cdots + a_n = a_{n+1} + \cdots + a_{n+r}$ given a certain r. Through the examination of a sequence of natural numbers that were both odd and even, he studied the possibility of sequence balancing and co balancing numbers. However, in regard to $n > 1,$ Not a single sequence balancing number could be found in the series that he was looking for. $a_k = k^n.$

RESEARCH METHODOLOGY

In the year 2011, Szak¬acs made the decision to investigate the possibility of multiplying balancing numbers n as natural numbers that satisfy the Diophantine equation. This was done in lieu of the additions that were used in the definitions of balancing and co balancing numbers. $1 \cdot 2 \cdot \cdots \cdot (n-1) = (n+1)(n+2)\cdots(n+r)$ shown that 7 is the only integer that can be used to multiply and balance for certain r. In addition to investigating sequence balancing numbers by making use of words from an arithmetic progression, T. Kovacs, K. Liptai, and P. Olajos [4] broadened the concept of balancing numbers to include arithmetic progressions. They dialed the number. $an+b$ an (a,b)- a number that is in equilibrium if $(a+b)+\cdots+(a(n-1)+b) = (a(n+1)+b)+\cdots+(a(n+r)+b)$ The statement is valid for any r in which the cop rime integers a > 0 and b > 0 are present. Specifically, they outlined the requirements that must be met for such numbers to be present. Within the scope of their investigation into specific instances of sequence balancing numbers, Komatsu and Szalay substituted binomial coefficients for the natural numbers that are included in the definition of balancing numbers.

The following is an example of how we might redefine balancing and co balancing numbers by making some modest algebraic alterations to the equations that describe balancing and co balancing numbers.

n is considered a natural number if it is a balanced number.

$$1+2+\cdots+m=O_1+O_2+\cdots+O_n \qquad \text{.................................(14)}$$

There is a natural integer m for which n is a co balancing number if and only if

$$1+2+\cdots+\ell=E_1+E_2+\cdots+E_n \qquad \text{.................................(15)}$$

As can be seen in binary numbers are grasped by a significant number of people.

$$B_1=\begin{bmatrix}0\\1\end{bmatrix}\ \widetilde{B}_1=\begin{bmatrix}1\\0\end{bmatrix}$$

$$B_2=\begin{bmatrix}0&0\\0&1\\1&0\\1&1\end{bmatrix}\ \widetilde{B}_2=\begin{bmatrix}1&1\\1&0\\0&1\\0&0\end{bmatrix}$$

$$B_3=\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\\0&1&1\\1&0&0\\1&0&1\\1&1&0\\1&1&1\end{bmatrix}\ \widetilde{B}_3=\begin{bmatrix}1&1&1\\1&1&0\\1&0&1\\1&0&0\\0&1&1\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

These binary matrices are able to be generated using the first order recurrence relation via their capabilities.

$$B_{n+1}=\begin{bmatrix}0_{2^n\times 1} & B_n\\ 1_{2^n\times 1} & B_n\end{bmatrix},\ \text{n>1}$$

with the first word

$$B_1=G_1$$

where

$$G_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

DATA ANALYSIS

Table 1 Galois group of the function

(u, v)	μ	x0	m
(−1, 3)	1.798158	653	870
(−2, 3)	-	651	868
(1, 3)	-	656	874
(2, 3)	-	658	876
(−1, 4)	1.780719	1054	1317
(−3, 4)	-	1051	1314
(1, 4)	-	1058	1322
(3, 4)	-	1061	1325

Table 2. Binary and grayscale numbers

Decimal order	Binary	Gray	Gray decimal
0	000000	000000	0
1	000001	000001	1
2	000010	000011	3
3	000011	000010	2
4	000100	000110	6
5	000101	000111	7
6	000110	000101	5
7	000111	000100	4
8	001000	001100	12
9	001001	001101	13
10	001010	001111	15

Gray arrays are helpful when it comes to the creation of recursive sequences of the third order. $\{u_n\}, \{v_n\},$ which, according to the beginning words, are either closely connected or lacunary (have syllables that are missing). a, b, c and $d, e, f,$ There are eight different pairs of recurrence relations, each of which is specified by and

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix} = \begin{bmatrix} \tilde{g}_{i\bullet} & g_{i\bullet} \\ g_{i\bullet} & \tilde{g}_{i\bullet} \end{bmatrix} [u_{k-3}, u_{k-2}, u_{k-1} v_{k-3}, v_{k-2}, v_{k-1}]^T \quad \text{.................................(16)}$$

in which $g_{i\bullet}$ and $\tilde{g}_{i\bullet}$ are the i-th rows of G_3 and $\tilde{G}_3$: and in accordance with. Have a go at it! When it comes to this topic, the two most important notions are locating patterns and acquiring the notation necessary to suggest innovative ideas.

"Good" sequences

n	'Good' sequences of length *n*	*an*
1	0	1
2	00,11	2
3	000,011,110,111	4
4	0000,0011,0110,1100,0111,1110,1111	7
5	00000,00011,00110,01100,11000,00111,01110,11100,01111,11110,11011,11111	12

The "good" figures, $a_n,$ it has been shown to be in accordance with the fourth order recurrence relation.: $a_n = a_{n-1} + a_{n-2} + a_{n-4}$.

Would you mind taking a look at some "better" sequences?, $\{b_n\}$ where it is required that each "one" (if it is present) be accompanied by two further "ones"; that is, in blocks of three lengths? (Note that the mathematical idea of exclusion-inclusion is being used in this particular instance.) The alternative is that Austin and Guy take into consideration binary sequences of length n, in which the "ones" only occur in blocks that are at least k in length. The components included inside this may be distinguishable by $a_n^{(k)}$, such that the "optimal" order is $\{a_n^{(2)}\}$, Once again, we witness how notation transforms from an artificial burden into a tool of cognition. Once again, this transformation takes place piece by piece. Please elaborate on that for me.

$$a_{k+n}^{(k)} = 1 + \tfrac{1}{2}(n+1)(n+2),\ 0 \le n \le k\ ? \quad \text{.................................}(17)$$

The issue known as the "lazy caterer's problem" has a solution, and that is the core polygonal numbers.

CONCLUSION

Galois is credited with developing a game-changing methodology for determining whether or not equations may be solved using radicals. It is believed that Galois was the inspiration for the naming of this paradigm, which would subsequently be known as the Galois theory. The idea of a Galois group, which describes the symmetrical connections that exist between the numerous roots of a polynomial problem, originated with him. He was the first person to ever propose the idea of a Galois group. Having access to the information that Galois groups provide on the structure and behavior of field extensions may be quite beneficial. Galois is recognized with producing a number of important contributions to the subject of mathematics, one of the most noteworthy of which was the creation of a criterion for judging whether or not a polynomial problem can be solved by use of radicals. Galois is also credited with making a number of other major contributions to the area. He proved that a polynomial equation may be solved by radicals if and only if the Galois group associated with the equation is of the kind that may be solved. This was his main contribution to the field of radical analysis. If the equation itself is one that can be solved, then this is the only alternative scenario in which it is conceivable for this to take place. According to these standards, there is a significant connection between the algebraic characteristics of a polynomial problem and the question of whether or not the problem can be solved. In particular, it was discovered that the issue could only be fixed if it had a certain amount of different variables.

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