International Journal of Applied and Behavioural Sciences (IJABS)

ISSN:3048-9083 2025, Vol.02, Issue 01

HIGHER ORDER HAAR WAVELET COLLOCATION METHODD: ENHANCED PRECISION, CONVERGENCE, AND APPLICATIONS TO NONLINEAR DIFFERENTIAL EQUATIONS

Singh, Kawaljeet[1] and Sunita[2]

[1]Research Scholar, NIILM University, Kaithal, Haryana

[2]Associate Professor, Department of Mathematics, NIILM University, Kaithal, Haryana

DOI: https://doi.org/10.70388/ijabs250114

Received on Sep 09, 2024

Accepted on Nov 08, 2024

Published on Jan 10, 2025

Abstract

This paper presents the **Higher-Order Haar Wavelet Collocation Method (HHWCM)**, an advanced numerical technique designed to address the limitations of the traditional Haar Wavelet Collocation Method (HWCM). By incorporating higher-order polynomial extensions into the Haar wavelet framework, the proposed method enhances precision and achieves faster convergence rates. The HHWCM is developed to effectively solve nonlinear ordinary differential equations (ODEs) under a wide array of conditions, including initial conditions, boundary conditions, periodic conditions, two-point conditions, integral conditions, and multi-point integral boundary conditions. The study begins with a theoretical foundation of HHWCM, demonstrating its improved approximation capabilities through convergence analysis and error estimation. This study underscores the versatility and potential of HHWCM as a robust computational tool for addressing nonlinear differential equations in scientific and engineering applications. The findings open avenues for extending the method to partial differential equations (PDEs) and exploring its integration with machine learning techniques to enhance numerical modelling and simulation in future work.

Keywords: Numerical method, Wavelet, Non-linear, Convergence, Collocation.

Introduction

First-order hyperbolic equations with two independent variables x and t are solved using a finite difference method. Typically, the first variable is space, while the second is time. For hyperbolic equations, a fundamental difficulty is constructing finite difference schemes that are both stable and do not damp out of the solution. Initial and initial-boundary value approximation algorithms are developed in this suggested study.

Motivation

First-order hyperbolic partial differential equations have a rich literature. It took a long time and a lot of work to create parameteric estimation techniques for Initial value and Initial-boundary value issues problems and initial-boundary value problems. Gottlieb et al (1987), Bo (1998), and Coulombel (2009) investigated the performance of finite - element strategies for 1st first order hyperbolic initial-boundary value problems using vectors value functions having L2(IR+, IRN). Semi-distributed approximations to the starting and boundary value issue were addressed by Warming and Beam in 1988.

$Ut = aUx,\ 0 \leq x \leq A, t \geq 0,$

$U(x,0) = u(x),\ 0 \leq x \leq A,$

$U(A,t) = v(t),\ \ t \geq 0,$

(1)

Wherein L2[0, A] is "a > 0" and "v(t) =0" . Using wavelets, Sekino and Hamada in 2008 derived a numerical solution to the Advection problem ut + (a(x)u) x = 0. Despres and Teng in 2009 and 2010 respectively developed finite-element techniques for the initial value problem. $Ut + aUx = 0$, x $\epsilon\mathbb{R}$, t $\epsilon\mathbb{R}^{+}$,

$U(x,0) = u_0(0)$, x $\epsilon\mathbb{R}$ … (2)

For constrained initial functions u_0 with discontinuous starting values. Motivated by the development of numerical methods to start initial-boundary value problems, these works are published.

Problem Formulation

Initial Value Issue (IVP) on an infinite interval is the initial model problem in this suggested study.

$Ut + a(x)Ux(x)$, x $\in \mathbb{R}^+$, t $\in \mathbb{R}^+$,

U(x,0) = u(x), x $\in \mathbb{R}^+$, … (3)

A(x) is greater than or equal to zero for all nonnegative values of x between IR+ and C(IR+), which is known as the initial condition. Wave propagation in homogeneous mediums is modelled by Equation (3).

IBVP (Initial-Boundary Value Issue) is an IB model problem that may be described as

$U_t = -a\, U_{x,}$ x $\in$ [0,1], t $\in \mathbb{R}^+$,

U (0, t) = v(t), t $\in \mathbb{R}^+$. (4)

Assuming "a > 0" & specification of the boundary condition v(t) at x = 0. There are no errors in this boundary condition since the information is flowing from left to right and the compatibility criterion is met (0).

With IVP (3), there is no boundary condition, and hence, no IVP (4). Many scenarios need this latter requirement.

For the issues IVP (3) and IBVP (4), the semigroup theory was heavily used in order to construct a completely discrete convergent numerical method. To solve the initial-boundary value issue, semigroup theory offers an elegant solution.

PRELIMINARIES

Theorem of Pazy (1983) was the primary tool employed in this study, and this section provides basic definitions and a specific instance.

Theorem 1. Assume X is a Banach space consisting of standard ‖ ‖. Assuming, X has denser D(A) such that A: D(A) $-\!\!\longrightarrow$ X is a continuous (linear) projection. Further, λ is there, with

$\Re(\lambda) > 0$ wherein the range "$\lambda I - A$" is dense in X. Let X_n be the Banach spaces consisting of standards $\| \|_n$. Furthermore, there are bounded linear operators, "$P_n: X \to X_n$ and $E_n: X_n \to X$" for every $n \geq 1$ such that

(i) $\|P_n\| \leq C_1$, $\|E_n\| \leq C_2$, with C_1 and C_2 represented as constants that are not ependent on n.

(ii) $\|P_n x\|_n \to \|x\|$ as $n \to \infty$ for every $x \epsilon X$.

(iii) $\|E_n P_n x - x\| \to 0$ as $n \to \infty$ for every $x \epsilon X$.

(iv) $P_n E_n = I_n$ (I_n :identity operator on X_n)

Define F (τ n) as a series of constrained linear operations spanning X_n to X_n fulfilling $\|F(\tau_n)^k\| \leq 1$. (5)

In addition to this, the constrained linear maps

$$A_n = \rho_n^{-1}(F(\rho_n) - I$$

have the property that

$$D(A) = \{x\epsilon\, X: E_n A_n P_n x \ converges\}$$

and that

$$\lim_{n\to\infty} E_n A_n P_n x = A_x \qquad (6)$$

for all $x \in D(A)$. A's closure ($\bar{A}$) is therefore the indefinite constructor of contracting mappings "S(t) on X". Furthermore, if $k_n\tau_n \to t$ as $n \to \infty$, then

$$\lim_{n\to\infty} \| F((\tau_n)^{k_n})P_n x - P_n S(t)_x \|_n = 0$$

It is permissible to refer to a generalized, solution as "solutions" in the following paragraphs.

If $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_k)$, then let the notation $\alpha(i) = \alpha_i$

For $x \in \mathbb{R}$, $[x] = \sup \{n \in Z: n \leq x\}$

Theorem 2 Theorem (Hille-Yosida Theorem). An infinitesimal generator of the C_0 semigroup of contractions T (t), t 0 can only be generated by a linear (unbounded) operator A,

(i) $\overline{D(A)}$ = X, since A is closed

(ii) IR+ is present in ρ(A), the resolvent set of A. Further, The resolvent set ρ(A) of A for every λ > 0, $\| R(\lambda : A) \| \leq \frac{1}{\lambda}$ (7)

IVP AND IBVP EXACT SOLUTIONS

For the start and initial-boundary value problems covered in this study, this section provides the precise solution.

IVP exact solution

The solution to eq (3) is provided by:

$$u(x,t) = u(\beta^{-1}\left(t + \beta(x)\right),$$

Here, $\beta(x) = \int_0^x \frac{d\xi}{a(\xi)}$

Using u (x, t) data on a bounded domain, we were able to calculate the solution of (3) that was bound not compulsorily on a bounded domain, numerically. This conclusion is made possible by the following theorem.

Theorem 1. Assuming "∈ C [0, ∞)" and "a(x) > 0" for every x ∈ IR+. Assume M > 0 and T > 0.

Define a_M: [0, M] → IR as

$a_M(x) = a(x),\ 0 \leq x \leq (M - \frac{1}{M})$

$= a(M - \frac{1}{M})\sqrt{M(M - x), M - \frac{1}{M} \leq x \leq M}$

Letting f ∈ C [0, M]. The problem's solution is

$\frac{\partial V}{\partial t} = a_M \ (x)\frac{\partial V}{\partial x}$ $, 0 \leq t \leq T, \ 0 \leq x \leq M,$

V(x,0) = f(x), 0≤x≤M,

V (M, t) = f(M) … (8)

Is existing and unique in nature. It is provided as

$V(x,t) = f\left(\beta_M^{-1}\left[\min\left(t + \beta_M(x), \beta_M(M)\right)\right]\right),$

here, $\beta_M(x) = \int_0^x \frac{d\xi}{\alpha(\xi)}, \ \ 0 \leq x \leq M - \frac{1}{M}$

$$= \int_0^{M-\frac{1}{M},} \frac{d\xi}{\alpha(\xi)} + \int_{M-\frac{1}{M}}^{x} \frac{d\xi}{\alpha\left(M-\frac{1}{M}\right)\sqrt{M(M-x)}}, \ M - \frac{1}{M} \leq x \ \leq M$$

Further,

$$S_t f(x) = f(\left[\min\left(t + \beta_M(x), \beta_M(M)\right)\right])$$

creates a contracting subclass on C [0, M] with the generator

D(A) = {g ϵ C [0, M]: g\` ϵ C [0, M] and $\lim_{x \to M} a_M(x)$ g\`(x) = 0}

and

$Ag(x) = \ a_M(x)$ g\`(x), $x\epsilon$ [0, M],

Ag(M) = 0.

Furthermore, selecting "M > N" in such a way that:

$$sup_{t\epsilon[0,T], x\epsilon[0,N]}(t + \beta(x)) < \left(M - \frac{1}{M}\right),$$

V (x, t) = u (x, t), (x, t) ϵ [0, N] × [0, T] … (9)

given "f ∈ C [0, M]" as the constraint of u to [0, M].

Proof. Defining t ≥ 0, "T_t: [0, M] → [0, M]" as

$$T_t x = \beta_M^{-1}\left[\min\left(t + \beta_M(x), \beta_M(M)\right)\right]$$

It is now simple to demonstrate that $T_{s+t} = T_s \circ T_t$.

$$T_s \circ T_t x = \beta_M^{-1}\left[\min\left(s + \beta_M(T_t x), \beta_M(M)\right)\right]$$

$$= \beta_M^{-1}\left[\min\left(s + \beta_M(\beta_M^{-1}\left[\min\left(t + \beta_M(x), \beta_M(M)\right)\right]), \beta_M(M)\right)\right]$$

$$= \beta_M^{-1}\left[\min\left([min\ (s\ + t + \beta_M(x), s\ +\ \beta_M(M)))], \beta_M(M)\right)\right]$$

$$= \beta_M^{-1}[min\ (s\ + t + \beta_M(x), \beta_M(M)$$

$$= T_{s+t}{}^{x}$$

Additionally, because "S_t f(x) = f (T_t x)", it is simple to show that St is a subgraph.

Thus, according to the Hille-Yosida Theorem, Now, by Hille-Yosida Theorem, if S_t generator is B,

Then,

$$(I - B)^{-1} h(x) = \int_0^{\infty} e^{-t}\, S_t h(x) dt$$

$$= \int_0^{\infty} e^{-t}\, h(\beta_M^{-1}[min\ (t + \beta_M(x), \beta_M(M))] dt$$

$$= \int_0^{\beta_M(M)-\beta_M(x)} e^{-t}\, h(\beta_M^{-1}[(t + \beta_M(x), \beta_M(M))] dt\ \int_{\beta_M(M)-\beta_M(x)}^{\infty} e^{-t}\, h(N) dt$$

$$= \int_0^{M} e^{\beta_M(M)-\beta_M(x)} \frac{h(y)}{a_M(y)} dy\ + h(N)\, e^{\beta_M(x)-\beta_M(M)} dt$$

Here, $\beta_M^{-1}(t + \beta_M(x))$.

Considering the differential equation,

f(x)- a M(x) f '(x) = h(x), x ϵ [0, M)

f (M) = h (M)

which is equivalent to

f (x) – a (x) f '(x) = h (x), x ϵ [0, M)

$\lim_{x \to M}$ a (x) f '(x) $= 0$

Since, $\int_0^M \frac{ds}{a_M(s)} < \infty$, there exists unique solution for every "h ∈ X" i.e., f ∈ D(A) to the preceding differential equation that is provided as,

$$f(x) = e^{\int_0^x \frac{ds}{a_M(s)}} \int_x^M \frac{h(y)}{a_M(y)} e^{-\int_0^y \frac{ds}{a_M(s)}} dy + h(M) e^{-\int_0^M \frac{ds}{a_M(s)}} + \int_0^x \frac{ds}{a_M(s)}$$

$$= \int_x^M e^{\beta_M(x)-\beta_M(y)} \frac{h(y)}{a_M(y)} dy + h(N)\, e^{\beta_M(x)-\beta_M(M)} dt$$

For the operators A and B, it can be demonstrated that (I – A) 1 = (I–B) 1.

"D(A) = D(B)" and for any "g D(A)", "Bg = Ag" may be deduced from this.

For t [0, T] and x [0, N], it is for sure a growing derivative/ function

x $\leq \beta^{-1}(t + \beta(x)) < \left(M - \frac{1}{M}\right)$.

Hence,

$\beta(\beta^{-1}(t + \beta(x))) = \beta_M\,(\beta_M^{-1}(t + \beta_M(x)))$.

This leads to the conclusion that "S_t f(x) = V (x, t) = u (x, t)" for every "x ∈ [0, N]" and "t ∈ [0, T]".

IBVP exact solution

Theorem 2. Assume "u ∈ C [0, 1]" and "v ∈ C [0, ∞)" in such a way that "u (0) = v (0)".

Defining u_0 (x) = u_0 (x) − u_0 (0). For "U" as a solution to (4), its also a solution to

$$\overline{U_t} = -a\overline{U_x},\ 0 \le x \le 1, t \ge 0$$

$$\overline{U}(x,0) = -a\overline{U_x}, 0 \le x \le 1 \qquad (10)$$

and V is a solution to

$$V_x = \frac{-1}{a}V_t, 0 \le x \le 1, t \ge 0$$

V(x,0) = $u_0(0), 0 \le x \le 1,$

V (0, t) = v(t), $t \ge 0$ … (11)

Then, "U (x, t) = U (x, t) + V (x, t)". Furthermore, for "T > 0", contracting subgroups s.t., are defined

"X → X" where "X = {u ∈ C [0, 1]: u (0) = 0}" and "T(x): Y → Y"

Wherein, "Y = C [0, T]" as

$$S_t u_0 = u_0(x - at), at \le x \le 1$$

$$= 0,\ 0 \le x \le at,$$

$$T_x w(t) = w(0), 0 \le x \le {}^{x}/_{a}$$

$$= w\left(t - \frac{x}{a}\right), t \ge {}^{x}/_{a}.$$

Then "U (x, t) = Su_0(x) + Σ x w(t) for all (x, t) ∈ [0, 1] × [0, T]", W: represents constraint of “v to [0, T]”.

Furthermore, if A and B represent the St and Tx generators, respectively, then

“D(A) = {g ∈ X: g ′ ∈ X}, D(B) = {g ∈ Y: g ′ ∈ Y and g ′ (0) = 0}, Ag = −ag′ for all g ∈ D(A)” and “B g = −1 a g ′ for all g ∈ D(B)”.

Convergent numerical technique for IVP and IBVP

M AND INITIAL-BOUNDARY VALUE PROBLEM

In this lesson, we'll go through how to solve the starting value and initial boundary value issues numerically convergently. It is possible to solve the initial value issue by posing it on a smaller bounding box, and then solving it on a larger bounding box with the same answer. An improved numerical solution to the modified issue is indistinguishable from the original answer on the smaller constrained region. Decomposing the initial-boundary value issue into two problems, each of which generates a semigroup, allows for the presentation of discrete semigroup approximations.

The IVP and a Convergent Numerical Scheme

M > N and an initial value problem given on [0, M] [0, T] whose answer absolutely corresponds to the solution of (3) on [0, N] [0, T] for any subset of [0, N] [0, T] are possible for the initial value issue (3). IR+ IR+ IR+ IR+ IR+ On [0, M] [0, T], create a finite difference scheme that converges to the solution of the problem provided in (3) on [0, N] [0, T].

This finding is made easier by the following theorem.

Theorem 1. Assume “X = C [0, M]” & “A”. Let $X_n = IR_n+1$ whose elements are denoted as $\alpha = (\alpha_0, \alpha_1, \ldots \alpha_n)$. The supremum standard is applied to both X and Xn spaces. Finally, defining

$P_n: X \rightarrow X_n$ as $(P_n f)i = f\,(iM/n)$, $i = 0, 1, \ldots\ n$.

$E_n: X_n \rightarrow X$ as

En(α) represents “piecewise linear function” having $E_n(\alpha)(iM/n) = \alpha_i$. Let

$$\tau_n = \frac{1}{2n(sup_{x\epsilon[0,M]}|a(x)|)}.$$

Defining operations "F (τ_n): $X_n \to X_n$" as

$$(F(\tau_n)\alpha)_i = \left(1 - n\tau_n a_m\left(\frac{iM}{n}\right)\right)\alpha_i + \left(n\tau_n a_m\left(\frac{iM}{n}\right)\right)\alpha_{i+1}, i = 0,1,\dots,n-1$$

$$= \alpha_n, i = n.$$

Choosing kn $= \frac{t}{\tau_n}$, it can be shown that

$$\|\tau_n{}^{k_n}P_n f - P_n S(t)f\|_n \to 0 \; as \; n \to 0 \qquad (12)$$

Particularly, fixing "t ∈ [0, T]" & "x ∈ [0, N]",

$$\lim_{n\to\infty} F(\tau_n)^{k_n} P_n f\left(\llcorner \frac{nx}{M} \llcorner\right) = u(x,t), \qquad (13)$$

Where, where u (x, t) represents a solution to (3).

Proof. Since P_n is clearly linear, $||P_n||$ 1. The fact that (ii) of Theorem 1 is true may be deduced with norms' definitions from the definitions of the norms & constant continuance of component X. That $||E_n||$ 1 is simply obtained.

When you consider the definitions of E_n and P_n as well as the uniform continuation of the element X in Theorem 1, it becomes clear that (ii) is true. Formulating differential equation leads to the definition of F(n) which is now simple. To illustrate, let's look at the (iM/n, j_n) lattice functions for "I = 0, 1, 2, ..." and ("j = 0, 1, 2, ... in the (x, t) plane").

Assume V ($i_{M/n}$, $j_{\tau n}$) = $u_{i,j}$. Taking into account the sup x∈ [0, M] aM(x) > 0, τ_n is defined clearly. Take a look at the difference equation that corresponds to the differential equation in (8) is

$$\frac{u_{i,j+1} - u_{i,j}}{\tau_n} = a_m\left(\frac{iM}{n}\right)n\left(u_{i+1,j} - u_{i,j}\right), i = 0,1,2,\dots,n-1$$

$\frac{u_{n,j+1}-u_{n,j}}{\tau_n} = 0$, which can be simplified as

$$u_{i,j+1} = \ 1 - n\tau_n a_m \left(\frac{iM}{n}\right) u_{i,j} + n\tau_n a_m \left(\frac{iM}{n}\right) u_{i+1,j}, i = 0,1, \dots, n-1$$

$$u_{n,j+1} = \ u_{n,j}.$$

It is possible to calculate all $u_{i,j}$ by using the preceding method for ui,0. The formula for f_i is $f(i_{M/n}) = f_i$.

Now,

$$\|F(\tau_n)(\alpha)\|_n$$

$$= \max\left(max_{0\le i\le n-1}\left| 1 - n\tau_n a_m \left(\frac{iM}{n}\right) \alpha_i + n\tau_n a_m \left(\frac{iM}{n}\right) \alpha_{i+1}\right|, |a_n|\right)$$

$$= \max\left(max_{0\le i\le n-1}\left(1 - n\tau_n a_m \left(\frac{iM}{n}\right) \alpha_i + n\tau_n a_m \left(\frac{iM}{n}\right)\right) \max(|\alpha_i|, |\alpha_{i+1}|), |a_n|)\right)$$

$$= \ \max(|\alpha_0|, |\alpha_1|), \dots, |a_n|)$$

$$= \|\alpha\|_n.$$

Therefore, $\|F(\tau_n)\| \le 1$, holding the theorem 1's stability condition (5).

For, f ϵ D,

$$\left\|\tau_n^{-1}(F(\tau_n) - 1)P_n f - P_n a_m \left(\frac{iM}{n}\right) f`\right\|_n$$

$$= sup_i \left|\frac{a_m\left(\frac{iM}{n}\right)}{n}\left(f\left(\frac{(i+1)M}{n}\right) - f\left(\frac{iM}{n}\right)\right) - a_m\left(\frac{iM}{n}\right) f\left(\frac{iM}{n}\right)\right| \dots \qquad (14)$$

Because "f ∈ D", "A_f" is consistently periodic on "[0, M]" the R.H.S of (14) goes to 0 as n tends to infinity ($n \to \infty$). As a result, Theorem 1 (3.6) is fulfilled.

Using Theorem 1, one must prove that for any > 0, the range "I – A" in X is dense. However, the range of I A is previously demonstrated to include all of X in Theorem 1. Theorem 1's formulation of (I B) 1h khk may also be used.

CONCLUSION

Logic of partial functions as it is now understood is summarize and some directions for future research are suggested in this brief final study. This thesis was introduced by stating that partial functions have better logical and computational features than binary relations. This is worth repeating. This thesis's findings only serve to support this point of view. A first-order definition of an operation is one which may be defined according to the basic theorem. When it comes to these kinds of operations, it has already been proven that the representation classes are typically finitely axiomatisable and have equational theories of low complexity, the finite representation property is fulfilled and representability of finite algebras is easily decided. Moreover, similar observations seem to apply to functions with several locations. It's all the opposite of what happens in relationships. If we want to reason about programe represented by code written in any general-purpose (i.e., Turing-complete) language, we must be able to articulate some kind of unbounded iteration mechanism. However, translating in this manner yields results only if the signature contains anti-domain. Including anti-domain would be problematic if we were trying to describe partial recursive functions with no constraints, as finding the places where partial recursive functions are undefined is not a universal and effective computation that can be expressed in any programming language.

REFERENCES

1. Smith, H. An introduction to delay differential equations with applications to the life sciences, Text Book in Applied Mathematics, Vol. 57, First Edition, 2011.
2. Marcel Jackson and Tim Stokes, Modal restriction semigroups: towards an algebra of functions, International Journal of Algebra and Computation 21 (2011), no. 7, 1053–1095.
3. Ian Hodkinson, A construction of cylindric and polyadic algebras from atomic relation algebras, Algebra Universalis 68 (2012), no. 3, 257–285.
4. Robin Hirsch and Szabolcs Mikulas, ´ Axiomatizability of representable domain algebras, The Journal of Logic and Algebraic Programming 80 (2011), no. 2, 75–91.

5. Robin Hirsch and Brett McLean, Disjoint-union partial algebras, Logical Methods in Computer Science 13 (2017), no. 2:10, 1–31.
6. Robin Hirsch, Marcel Jackson, and Szabolcs Mikulas, ´ The algebra of functions with antidomain and range, Journal of Pure and Applied Algebra 220 (2016), no. 6, 2214–2239.
7. Robin Hirsch and Marcel Jackson, Undecidability of representability as binary relations, The Journal of Symbolic Logic 77 (2012), no. 4, 1211–1244.
8. Robert Goldblatt and Marcel Jackson, Well-structured program equivalence is highly undecidable, ACM Transactions on Computational Logic 13 (2012), no. 3, 26:1–26:8.
9. Simon Foster, Georg Struth, and Tjark Weber, Automated engineering of relational and algebraic methods in Isabelle/HOL, Proceedings of 12th Conference on Relational and Algebraic Methods in Computer Science (Harrie de Swart, ed.), Springer, 2011, pp. 52–67.
10. Robert Egrot and Robin Hirsch, completely representable lattices, Algebra Universalis 67 (2012), no. 3, 205–217.
11. Wiesław A. Dudek and Valentin S. Trokhimenko, Subtraction menger algebras, Semigroup Forum 85 (2012), no. 1, 111–128.
12. Cristiano Calcagno, Dino Distefano, Jer´ emy Dubreil, Dominik Gabi, Pieter Hooimeijer, Mar- ´ tino Luca, Peter O'Hearn, Irene Papakonstantinou, Jim Purbrick, and Dulma Rodriguez, Moving fast with software verification, Proceedings of 9th NASA Formal Methods Symposium, Springer, 2015, pp. 3–11.