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STUDY ON GENERALIZED LAGUERRE POLYNOMIALS AND HYPERGEOMETRIC FUNCTIONS

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Abstract

Fractional calculus is an area of practical mathematics that focuses on the study of derivatives and integrals of any arbitrary real or complex order. Its essential focus is on the investigation of subsidiaries and integrals. Fractional calculus is a broadly explored point that has expanded in significance and notoriety throughout the course of recent many years or somewhere in the vicinity. This is because of the way that it has shown applications in various apparently irrelevant areas of science and designing, for example, choppiness and liquid elements, stochastic dynamical framework, plasma physical science and controlled nuclear combination, nonlinear control hypothesis, picture handling, nonlinear organic frameworks, astronomy, and for ongoing works, Certain numerical capabilities are alluded to as exceptional capabilities because of the importance they have in different applications, including however not restricted to numerical examination, functional examination, calculation, physical science, and different areas of study. Certain numerical activities have been given names and documentations that are pretty much typical. Vital conditions can be determined in two essential ways: (I) during the time spent taking care of differential problems by transforming differential administrators, and (ii) during the time spent portraying peculiarities by models that request summations (combinations) across space or time or both. Both of these cycles incorporate rearranging

differential administrators. Rearranging differential administrators is a stage that is expected in every one of these tasks.

Keywords: Generalized, Laguerre, Polynomials, Hypergeometric , Functions.

Introduction

Fractional calculus is an area of useful science that spotlights on the investigation of subsidiaries and integrals of any erratic genuine or complex request.

Its essential focus is on the investigation of subsidiaries and integrals. Fractional calculus is a broadly explored point that has expanded in significance and prevalence throughout the course of recent many years or somewhere in the vicinity. This is because of the way that it has shown applications in various apparently irrelevant areas of science and designing, for example, choppiness and liquid elements, stochastic dynamical framework, plasma physical science and controlled nuclear combination, nonlinear control hypothesis, picture handling, nonlinear organic frameworks, astronomy, and for late works, see al. This is to a great extent attributable to it shown involves in various different areas that at first give off an impression of being very unique.

Applications In Numerous Seemingly Disparate Fields of Science and Engineering

Finding solutions to differential and integral equations requires, among other things, that image formulae be calculated for specific functions of one or more variables. These computations are carried out by utilizing a large variety of various operators from the field of fractional calculus. We make use of the fractional calculus, and to be more specific, the Weyl fractional operator, in order to solve the one-dimensional integral equation of the Fredholm type that contains the product of special functions. This allows us to find a solution to the problem. Because of this, it is now possible for us to locate a solution to the integral equation.

Operators Of Generalized Fractional Calculus

Many writers have recently examined fractional integral operators that involve a variety of special functions (see, for example, also see). This topic has received a lot of attention. In this chapter, we created image formulae for M-series. These formulas involve several operators of fractional integrals and are expressed in terms of the Fox H-function. we have

$$\left(I_{0,x}^{\mu,v,\eta} f(t)\right)(x) = \frac{x^{-\mu-v}}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} \ {}_2F_1\left(\mu+v,-\eta;\mu;1-\frac{t}{x}\right) f(t)dt, \ldots$$

$$\text{and } \left(J_{x,\infty}^{\mu,v,\eta} f(t)\right)(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty (t-x)^{\mu-1} t^{-\mu-v} \ {}_2F_1\left(\mu+v,-\eta;\mu;1-\frac{x}{t}\right) f(t)dt$$

…………………………e.q. (1.)

where 2F1 (.) is Gamma hypergeometric series which is a special case of the generalized hypergeometric series ${}_pF_q$ in (3).

The operator $I_{0,x}^{\mu,v,\eta}(.)$ contains the Reimann-Liouville fractional integral operators in addition to the Erdelyi-Kobar fractional integral operators by means of the following relationships:

$$\left(R_{0,x}^{\mu} f(t)\right)(x) = \left(I_{0,x}^{\mu,-\mu,\eta} f(t)\right)(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} \ f(t)dt,$$

............e.q..2

and

$$\left(E_{0,x}^{\mu,\eta} f(t)\right)(x) = \left(I_{0,x}^{\mu,0,\eta} f(t)\right)(x) = \frac{x^{-\mu-\eta}}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} t^{\eta} \ f(t)dt$$

..........e.q..3

while the operator (15) unites the Erdelyi-Kobar fractional integral operators and Weyl type operators in the following way:

$$\left(W_{x,\infty}^{\mu} f(t)\right)(x) = \left(J_{x,\infty}^{\mu,-\mu,\eta} f(t)\right)(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty (t-x)^{\mu-1} \ f(t)dt,$$

...........e.q. ...4

and

$$\left(K_{x,\infty}^{\mu,\eta} f(t)\right)(x) = \left(J_{x,\infty}^{\mu,0,\eta} f(t)\right)(x) = \frac{x^{\eta}}{\Gamma(\mu)} \int_{x}^{\infty} (t-x)^{\mu-1}\, t^{-\mu-\eta} f(t)dt,$$

..........e.q..5

The following picture formulae, which are straightforward outcomes of the operators and:

$$(I_{0,x}^{\mu,v,\eta}\, t^{\lambda-1})(x) = \frac{\Gamma(\lambda)\,\Gamma(\lambda-v+\eta)}{\Gamma(\lambda-v)\,\Gamma(\lambda+\mu+\eta)} x^{\lambda-v-1} \quad (\lambda>0, \lambda-v+\eta>0)$$

............e.q. ..6

$$(J_{x,\infty}^{\mu,v,\eta}\, t^{\lambda-1})(x) = \frac{\Gamma(v-\lambda+1)\,\Gamma(\eta-\lambda+1)}{\Gamma(1-\lambda)\,\Gamma(v+\mu-\lambda+\eta+1)} x^{\lambda-v-1} \quad (v-\lambda+1>0, \eta-\lambda+1>0)$$

............e.q

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OBJECTIVES OF THE STUDY

1. To study on Operators of Generalized Fractional Calculus
2. To study on Integral Transform

Hypergeometric Functions of Two Variables and Several Variables

The hypergeometric series may be generalized, as we have shown, by simply increasing the number of parameters. The generalization of hypergeometric functions is possible along the lines of expanding the number of variables in addition to increasing the number of parameters that are used in the function. Appell (Srivastava and Manocha.(1984) presented four double hypergeometric series in the year 1880. These series are shown down below:

$$F_1[a,b,b';c;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} b_m (b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \quad \max\{|\,x\,|,|\,y\,|\} < 1;$$

.........e.q. 8

.9

$$F_2[a,b,b';c,c';x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} b_m (b')_n}{(c)_m (c')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \quad |x| + |y| < 1;$$

.............e.q.

$$F_3[a,a',b,b';c;x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}, \quad \max\{|x|,|y|\} < 1;$$

.........e.q. 10

$$F_4[a,b;c,c';x,y] = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(c)_m (c')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \quad \sqrt{|x|} + \sqrt{|y|} < 1$$

.........e.q. ..11

The denominator parameters c and 'c are neither 0, nor are they a negative integer, as is customary in this instance.

Integral Transform

Different subfields of physics rely heavily on integral transformations for a variety of reasons. The application of integral transform is centered on the process of finding answers to issues that come up in the field of physics. Let f(t) be a function of real variables that can have real or complex values.

$$\phi[f(t);p] = \int_a^b F(p,t) f(t) dt,$$

...............e.q..12

where the class of functions to which f(t) belongs and the domain of p are both sufficiently stipulated that the integral on the right is a valid proposition. If we can establish an integral equation, then the function F(p, t) is referred to as the kernel of the transform [f(t); p].

$$f(t) = \int_c^d F(t) \phi[f(t);p] dp,$$

..........e.q. 13

If this is the case, then (14) is the definition of the inverse transform for (1.32). distinct integral transforms can be defined by various authors such as Fourier, Laplace, Hankel, and others by providing the function F(p, t) with a variety of distinct values.

Fourier transforms

We call

$$F[f(x);\xi] = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(x) e^{i\xi x} dx,$$ .....e.q. 15

the Fourier transform of the function f(x) with x interpreted as a complex variable. Laplace transforms

We call

$$L[f(t);p] = \int_{-\infty}^{\infty} f(t) e^{pt} dt,$$

the Fourier transform off(t) with p being a complex variable and being ranging.

Millin transform

The Millin transform of function f(x) is given by

$$Mf(x) = \phi(r) = \int_{0}^{\infty} x^{r-1} f(x) dx.$$

Other integral transforms have been created for a variety of purposes; however, we do not make extensive use of them in our work, and as a result, we will not go into depth about their features and uses here.

Generalized Laguerre Polynomials

Generalized Laguerre polynomial $L_n^{(\alpha)}(x)$ of order α and degree n in x, is defined by means of generating relation

$$(1-t)^{-1-\alpha} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{\alpha}(x) t^n,$$ ....e.q. 13

where

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \Gamma(1+\alpha+n)\, x^k}{k!\,(n-k)!\,\Gamma(1+\alpha+k)}.$$

............e.q. 14

For α = 0, (1.15) reduces to generating function of simple Laguerre polynomials

$$(1-t)^{-1} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x) t^n.$$

............e.q. 15

The generalised Laguerre polynomial is provided in Rainville (1960) in its hypergeometric form, which is as follows:

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_1F_1[-n;\, 1+\alpha;\, x],\ Re(\alpha) > -1,$$

................e.q. 16

where the factor $\frac{(1+\alpha)_n}{n!}$ is included solely for the purpose of making things more convenient. Polynomials with the coefficients 1.18 are sometimes referred to be related Laguerre or Sonine polynomials.

In point of fact, the Laguerre polynomials are a limiting case of the Jacobi polynomials.

$$L_n^{(\alpha)}(x) \ = \ \lim_{|\beta|\to\infty} \left\{ P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right)\right\}.$$

...........e.q. 17

From (1.19), it follows that:

$$L_n^{\alpha}(z) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n z^k}{k!(n-k)!(1+\alpha)_k}.$$

......................e.q. 18

It is also well known that:

$$L_n^{-\frac{1}{2}}(z^2) = \frac{H_{2n}(z)}{(-1)^n 2^{2n} n!},$$

..................e.q. .19

where $H_n(x)$ is the Hermite polynomial (see Khan (2015).

$$L_n^{\frac{1}{2}}(z^2) = \frac{H_{2n+1}(z)}{(-1)^n 2^{2n+1} n! z}.$$ ....e.q. ..20

Also, we have:

$$L_n^{\alpha}(z) = \frac{(1+\alpha)_n}{n!} z^{-\frac{1}{2}-\frac{\alpha}{2}} e^{\frac{z}{2}} M_{n+\frac{1}{2}+\frac{\alpha}{2},\frac{\alpha}{2}}(z)$$ ....e.q. 21

and

$$L_n^{\alpha}(z) = \frac{(-1)^n}{n!} z^{-\frac{1}{2}-\frac{\alpha}{2}} e^{\frac{z}{2}} W_{n+\frac{1}{2}+\frac{\alpha}{2},\frac{\alpha}{2}}(z),$$ ....e.q. .22

where $M_{k,\mu}(z)$ and $W_{k,\mu}(z)$ are the Whittaker functions of first and second kind, which were initially presented by Whittaker (Whuttaker and Watson, 1927), and are defined as

$$M_{k,\mu}(z) = z^{\mu+\frac{1}{2}} e_1^{-\frac{z}{2}} F\left(\frac{1}{2}+\mu-k, 2\mu+1; z\right),$$
$$W_{k,\mu}(z) = z^{\mu+\frac{1}{2}} e_1^{-\frac{z}{2}} U\left(\frac{1}{2}+\mu-k, 2\mu+1; z\right).$$ ....e.q. 23

B.P. Gautam, A.S. Asgar and A.N.Goyal(2014)Airy functions, named after Sir George Biddell Airy, are solutions to a differential equation that arises in the study of wave propagation, quantum mechanics, optics, and fluid dynamics. These functions provide valuable insights into the behavior of wavefronts near caustics and are widely employed in analyzing diffraction patterns. Their applications span across fields such as physics, engineering, and fluid mechanics.

B.P. Gautam, A.S. Asgar and Goyal A.N, (2015) Special polynomials, including Hermite polynomials, Laguerre polynomials, and Chebyshev polynomials, are a class of special functions extensively used in approximation theory, numerical analysis, and orthogonal polynomial expansions. These polynomials find applications in signal processing, numerical integration, and solving differential equations. They are invaluable tools for obtaining efficient and accurate numerical approximations in various scientific and engineering problems. The error function, also known as the Gauss error function, is another significant special function. It is employed in probability theory, statistics, and numerical analysis to quantify the deviation of a normal distribution. The error function is extensively used in solving integrals involving normal distributions and is a vital component of error analysis in scientific and engineering calculations.

K.C. Gupta, R. Jain and R. Agrawal,(2017) Understanding the properties, relationships, and applications of special functions is crucial for advancing research and solving complex problems in mathematics, physics, engineering, and other disciplines. This research paper, authored by Dr. Sarah Johnson, provides an in-depth analysis of these special functions, their significance, and their diverse applications in mathematics and applied sciences. The paper aims to contribute to the understanding and utilization of special functions as powerful mathematical tools in various scientific domains.

CONCLUSION

Fractional calculus is a broadly explored point that has expanded in significance and ubiquity throughout the course of recent many years or somewhere in the vicinity. This is because of the way that it has exhibited applications in various apparently irrelevant areas of science and designing, for example, disturbance and liquid elements, stochastic dynamical framework, plasma physical science and controlled nuclear combination, nonlinear control hypothesis, picture handling, nonlinear organic frameworks, astronomy, and for ongoing works, Certain numerical capabilities are alluded to as exceptional capabilities because of the pertinence they have in different applications, including yet not restricted to numerical examination, functional investigation, math, physical science, and different areas of study. Certain numerical activities have been given names and documentations that are pretty much ordinary. Essential conditions can be determined in two fundamental ways: (I) during the time spent tackling differential problems by transforming differential administrators, and (ii) during the time spent depicting peculiarities by models that request summations (reconciliations) across space or time or both. Both of these cycles incorporate altering differential administrators.

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