International Journal of Applied and Behavioural Sciences (IJABS)

ISSN:3048-9083 2025, Vol.02, Issue 01

METHODS FOR SOLVING AN INTEGRAL EQUATION NUMERICALLY AND ANALYTICALLY

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DOI: https://doi.org/10.70388/ijabs250116

Received on Sep 06, 2024

Accepted on Nov 01, 2024

Published on Jan 10, 2025

Abstract

Integral equations, which depict the connections between functions as integrals, are very important in a variety of scientific and technical domains. It is vital to solve integral equations, both numerically and analytically, in order to comprehend and forecast occurrences that occur in these areas. The purpose of this work is to investigate several approaches to the resolution of integral equations. We concentrate on numerical techniques, such as numerical quadrature and iterative methods, as well as analytical approaches, such as the separation of variables, Green's functions, and integral transforms. In this section, we analyze the benefits and drawbacks of each technique, focusing on how each one may be applied to various kinds of integral equations and problem domains.

Keywords: integral equation, analytically, equation numerically

Introduction

Integral equations are often required in the process of resolving a variety of issues requiring initial values and boundary conditions that include ordinary and partial differential equations. These are the kinds of issues that call for a diverse collection of differential equations. Consequently, issues in mathematical physics involving beginning values and boundary values may be solved by first reducing these problems into appropriate integral equations in order to identify their solutions. This allows for the problems' solutions to be determined. Integral equations, much like any other type of mathematical equation, can have either a linear or non-linear form, depending on the specifics of the problem at hand. There are a lot of problems in science and engineering that require integral equations.

It is possible to convert a wide variety of initial value issues and boundary value problems to integral equations based on Volterra and Fredholm, respectively. Integral equations can be found in a wide variety of applications throughout the fields of applied mathematics and physics. They provide an effective method for resolving a range of practical issues in a variety of contexts. The fact that all of the conditions that describe the starting value issues or boundary value problems for a differential equation can frequently be condensed into a single integral equation is one of the most apparent justifications for utilizing the integral equation rather than the differential equation. This is one of the reasons why the integral equation is preferred over the differential equation. While still dealing with partial differential equations, this technique brings the complexity of the problem down to a more manageable level. For example, a boundary value problem for a two-variable partial differential equation may be expressed as a one-variable integral equation with an unknown function if one of the variables is replaced with "unknown function." It is a significant accomplishment in its own right to condense what could otherwise be a complex mathematical representation of a physical situation into a single equation. This is because reducing anything to its simplest form requires significant effort. On the other hand, the benefits of integrating similarities rather than differences are elaborated upon in the following sentences.

Some of these benefits are made possible by the fact that integration is a smooth process. This characteristic has major consequences whenever approximate answers are being sought. Those who are looking for a precise solution as well as those who are forced to settle for an approximation may often profit from the formulation of an integral equation. This is because integral equations are utilized in the modeling of the dynamics of integrating functions. The reason for this may be seen in the previous sentence. Integral equations have been a topic of discussion among mathematicians for the better part of the last century, which has allowed for the development of their respective theories. There is a substantial body of research on integral equations of the second sort, but the research on integral equations of the first kind is far more limited (see. Mikhlin (1957), Muskhelishvili (1963), Kanwal (1989), Porter and Stirling (1990), and Atkinson(1997)). Exact solutions can be found for integral equations of the first kind when certain particular forms of the kernel are used. The integral equations are considered singular in the sense that the kernels of each of them contain singularities of some specialized variety. Depending on the circumstances, the singularity could have a mild or powerful presence. When dealing with kernels that only have a weak singularity, the integral that corresponds to them is typically defined using the Riemann sense. However, when dealing with strongly singular kernels, the integrals need to be formulated in an acceptable manner in order for them to make sense from a mathematical perspective. For instance, the integral of Cauchy is defined in terms of the Cauchy principal value, whereas the integral of hypersingular is defined in terms of the Hadamard finite part. As a result of this, the study of the various strategies that can be used to solve singular integral equations has become an important field. Analytical and numerical approaches to solving integral equations both need to be studied in depth in order to provide a comprehensive understanding of integral equations and their solutions.

Occurrences of Integral Equations and Examples

There are numerous issues in mathematical physics that require integral equations to be solved. It is feasible to show that the solution of certain integral equations is equivalent to the solution of problems involving beginning values and boundary values that are provided by ordinary differential equations (ODE) and partial differential equations (PDE). This is something that can be done via the use of certain integral equations. In this piece, we will talk about the appearance of integral equations in a basic problem from classical mechanics, as well as in ordinary and partial differential equations. This topic will also be covered in the previous section. There is a good chance that the well-known mathematician Henrik Abel (1802-1829) was the one who first developed integral equations. In order to determine the form of a smooth curve that has a specific end point, along which a pstudy slides under the force of gravity in a given amount of time, which is independent of the starting position of the pstudy, he first reduced it to the issue of solving an integral equation. This allowed him to determine the shape of the curve without regard to the starting location of the pstudy. The name "Abel integral equation" has been given to this particular integral equation by the general public.

OBJECTIVES

1. To investigate the nonlinear integral equation as well as the integro-differential equation utilizing numerical techniques
2. To investigate and gain an understanding of the historical context around the Fredholm integral equation of the Second Kind.

Abel's problem First,

In this piece, we'll investigate a problem from classical mechanics in which we attempt to determine the amount of time necessary for a pstudy to go from any point (X, Y) on a fixed curve in the vertical xy plane, while being subjected to the force of gravity, to the origin, which is the point at which the curve is at its lowest point. We'll do this by assuming that the pstudy is moving at a constant speed. When we substitute m for the mass of the pstudy and x = (y) for the equation of the smooth curve, where is a function that can be differentiated based on y, we are able to derive the energy conservation equation, which is described in the following way:

$$\frac{1}{2}mv^2 + mgy = mgY \qquad \ldots\ldots\ldots\ldots..1$$

Where v is the speed of the pstudy at the location (X, Y) at time t, under the assumption that the pstudy begins falling from rest at time t = 0 from the point (X, Y), and g is the acceleration due to the force of gravity. The assumption is that the pstudy starts falling from rest at time t = 0 from the point (X, Y).

The relationship (1), which may be expressed as

$$\frac{ds}{dt} = -\left[2g(Y-y)\right]^{\frac{1}{2}} \qquad \ldots\ldots\ldots\ldots.2$$

Using the negative sign in the square root calculation because, throughout the course of the fall of the pstudy, the arc-length s(t) that is measured from the origin to the point (x, y) becomes shorter as time t progresses. Calculating the radial distance between the origin point and the coordinates (x, y) is the first step in accomplishing this goal. Putting the formula into use

$$\frac{ds}{dt} = \left[1 + (\Psi'(y))^2\right]^{\frac{1}{2}} \qquad \ldots\ldots\ldots\ldots.3$$

Where $\Psi'(y) = \frac{d\Psi}{dy}$, we can express (1) in the form

$$\frac{dy}{dt} = \frac{dy}{ds}\frac{ds}{dt} = -\left[\frac{2g(Y-y)}{1+(\Psi'(y))^2}\right]^{\frac{1}{2}} \qquad \ldots\ldots\ldots\ldots.4$$

and, this on integration, gives

$$\int_Y^0 \left[\frac{1+(\Psi'(y))^2}{2g(Y-y)}\right]^{\frac{1}{2}} dy = -\int_0^T dt = -T \qquad \ldots\ldots\ldots\ldots.5$$

where T is the total amount of time it takes for the pstudy to descend all the way from the point (X, Y) back to the initial position (0, 0). Written words on a page

$$\phi(y) = \left[\frac{1+(\Psi'(y))^2}{2g}\right]^{\frac{1}{2}} \qquad \ldots\ldots\ldots\ldots.6$$

the relation (5) can be written as

$$\int_0^Y \frac{\phi(y)}{(Y-y)^{\frac{1}{2}}} dy = T = f(Y), \text{say}, Y > 0. \qquad \ldots\ldots\ldots\ldots.7$$

Take note that the value of f(0) is equal to zero. If the shape of the curve x = (y), and therefore the function (y), is known, then we find that the time of fall of the pstudy, T, can be fully calculated by using the formula (7). This is the case if we know the function (y). Because of this, we are able to draw the conclusion that the time that the pstudy will descend can be precisely determined.

However, if we concentrate on Abel's initial task, which was to calculate the form of the curve given just the time of fall T(= f(Y)), then connection (7) becomes an integral equation for the unknown function y, and it is for this reason that it is referred to as Abel's integral equation or simply Abel's integral. The relationship between the illusive function and the decreasing time is defined by this integral equation. The equation for the Abel integral looks like this in its simplest version, which is as follows:

$$\int_0^x \frac{\phi(t)}{[h(x)-h(t)]^\alpha} dt = f(x), x > 0(f(0) = 0) \qquad \text{………….8}$$

where h(x) is a function that increases gradually and is a real constant in the range from 0 to 1 and always remains the same. This constant exists in the range from 0 to 1. Equations (7) and (8) have been brought to our notice, and it has been stated that these equations are linear Volterra integral equations of the first class. These equations are examples of integrals that are weakly singular. If we substitute into the equation (8) the values of $h(x) = x^2, h(x) = \sin x (0 < x < \frac{\pi}{2}), h(x) = 1 - \cos x (0 < x < \frac{\pi}{2}),$ Then, we arrive to certain unique integral equations based on Abel's work. In the year 1826, Abel made the discovery that led to the equation (7), and as a result, the year 1826 is often considered to be the birth year of the integral subject equation.

Fredholm Integral Equation

Integral equations based on the Fredholm method are easily identifiable by characteristics such as the presence of constrained limits of integration. The condition that "if the kernel K(x, t) is continuous in the area [a, b]" holds if and only if the kernel K(x, t) is continuous in the region [a, b]"; in other words, the condition that "if the kernel K(x, t) is continuous in the region [a, b]" holds if and only if the singularities of the kernel are such that the double integral is satisfied. If the singularities of the kernel fulfill the double integral, then the condition "if the kernel K(x, t) has singularities such that the double integral is satisfied" may also be true.

$$\int_a^b \int_a^b |K(x,t)|^2 \, dx dt$$

exists and is finite, then the integral equation that corresponds to it is known as a regular Fredholm type integral equation. If the variable being integrated over is also finite. The standard representation of the Fredholm integral equation for the first type is as follows:

$$\int_a^b K(x,t)\phi(t)dt = f(x),\ a \leq x \leq b,$$

and the Fredholm integral equation of the second kind is given by,

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\phi(t)dt,\ a \leq x \leq b,$$

where is a parameter that can either be a real value or a complex value. In the context of practical use, a physical amount is denoted by the symbol ''. Even though there is alternative type of equation that might sometimes seem to be more user-friendly, the most common way that The equation (1.3.1) represents the second class of Fredholm integral equations. The name "Fredholm integral equation" describes this specific kind of equation.

$$\mu\phi(x) = f(x) + \int_a^b K(x,t)\phi(t)dt,\ a \leq x \leq b,$$

where it can be shown that 1 = is equal to 1. Also, there is no reduction in generality as a result of the incorporation of into both the kernel and the forcing term. Setting = 0 yields the first kind Fredholm integral equation, which is a benefit of the representation (1.3.4), which may be found in the next sentence.

Equation of Integrity with Respect to Volterra It is stated that an integral equation is of the Volterra type if either one or both of The integral equation that corresponds to an integration problem will have limits of integration that are either well-known functions of x or just the value x itself. If this is the case, then the integral equation does not include any unknown variables or terms. The equation for the first kind of Volterra integral looks like this when written out in its full form:

$$\int_a^x K(x,t)\phi(t)dt = f(x),\ a \leq x \leq b,$$

and the corresponding second kind is

$$\phi(x) = f(x) + \lambda \int_a^x K(x,t)\phi(t)dt,\ a \leq x \leq b,$$

or alternatively,

$$\mu\phi(x) = f(x) + \int_a^x K(x,t)\phi(t)dt, \; a \leq x \leq b.$$

When is set equal to zero, the first type of the Volterra integral equation is obtained.

CONCLUSION

At the end of this discussion, we have investigated a variety of approaches to solving integral equations, including both analytical and numerical approaches. The use of numerical techniques offers effective tools for solving integral equations in situations when analytical solutions are either intractable or practically unfeasible. Numerous techniques, including numerical quadrature and iterative methods, provide a wide range of methodologies that may be used to approximate answers with a high degree of precision. Analytical approaches, on the other hand, provide more in-depth insights into the mathematical structures that lie underneath integral equations. Methods like as the separation of variables, Green's functions, and integral transformations make it possible to arrive at explicit solutions and make it easier to comprehend physical events. The particular properties of the integral equation, the amount of precision that is sought, and the computing resources that are available all need to be taken into consideration when selecting a technique. Researchers are able to handle a broad variety of issues in a variety of fields if they have a solid knowledge of these approaches and are able to successfully use them. This helps researchers advance our understanding of complicated natural phenomena and systems.

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