NIILM University International Journal of Applied and Behavioural Sciences (IJABS)

ISSN:3048-9083 2025, Vol.02, Issue 01

THEORETICAL DERIVATION OF MULTIPLE FIELD CONCEPTS IN CONTEMPORARY ALGEBRA

Pooja [1] and Jakhar, Manjeet Singh[2]

[1]Research Scholar, Department of Physics, NIILM University, Kaithal, Haryana

[2]Associate Professor, Department of Physics, NIILM University, Kaithal, Haryana

DOI: https://doi.org/10.70388/ijabs250118

Received on Sep 09, 2024

Accepted on Nov 05, 2024

Published on Jan 30, 2025

Abstract

Moreover, within the realm of field theory, several advanced concepts emerge, such as field automorphisms, which are isomorphism's from a field to itself preserving the field operations. These automorphisms play a crucial role in Galois theory, providing insights into the structure of field extensions and the solvability of polynomial equations by radicals. Furthermore, the notion of field extensions and their degrees elucidates the relationships between fields, paving the way for deeper explorations into algebraic structures. The algebraic structure of vector spaces and other mathematical objects is defined by the features of the fields that underlie them. The behaviour of vector addition and scalar multiplication is governed by these fields. As a result, fields are an essential part of contemporary algebra and have many applications in many areas of mathematics and beyond. Their mathematical formulation and study include a wide range of notions, from basic qualities to complex theories.

Keywords: contemporary, algebra, vector, galois theory.

Introduction

Clusters, circles, and areas to explore. Groups. Groups. During this period of half a year, we will talk about various distinct algebraic structures, the three most important of which are rings, fields, and sets, along with some of their more nuanced versions.

To begin, let's look at some definitions and then move on to some examples. At this time, we will not be proving anything; rather, this information will be offered in the next chapters when we continue to examine these structures in more detail.

A commercial about the notation. For all of the other numbers, we would use the standard notation. The comprehensive compilation of all real numbers , $\{0, 1, 2, \ldots\}$ "N" has a label. Counting all integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is labelled Z (for numbers, whole number German). The set of logical numbers, namely type numbers $\frac{m}{n}$ Since both m and n are integers that are not zero, the equation will be referred to as a "quotient" by the mathematical community. R is the symbol for all "real" numbers, including those that are positive, those that are negative, and 0 itself. In addition, the spectrum of complex numbers, often known as type numbers $x + iy$ Yes, x and y do in fact exist, and $i^2 = -1,$ is represented by C.

Investigating the spheres, fields, and clusters of objects. The three primary types of algebraic structures that will be discussed during the course of this semester are fields, rings, and groups. Additionally, various variations on these basic concepts will also be addressed. There is a comprehensive discussion of each of the three primary categories in the chapters that correspond to them: 2, 3, and 4. Let's have a look at some definitions and illustrate them with some instances before we get started. The evidence will be saved for subsequent chapters, when we will examine those structures in more detail; for the time being, we shall operate on the assumption that they do exist.

Note: this is just a quick aside. In order to represent all of the different kinds of numbers, the standard notation will be used. The letter N denotes the collection of all numbers, which may be regarded as fundamental or fundamental components. Z is the sign that is used to indicate the set of numbers {..., −2, −1, 0, 1, 2,...} in German. A collection of all the numbers that are capable of being stated as rational expressions, that is, those that have the form $\frac{m}{n}$ In situations in which both m and n are integers that are not zero, the symbol "Q" (which stands for "quotient") is often used. It is referred to as the "set of all real numbers," or simply R, and it is a collection of integers that contains all positive numbers, all negative numbers, and zero of all possible values. Complex numbers, often known as C, are a collection of real numbers that may be stated as x + iy, where i 2 = -1. This collection includes all real numbers x and y.

Performing operations on collections

More explanation is given in the "Sets" section (appendix section A.2).

Addition, subtraction, multiplication, division, negation, and dealing with the reciprocals, powers, and roots of real numbers R are the typical mathematical operations that are performed while working with real numbers R.

Binary operations, often known as functions, may be represented by examples such as addition, subtraction, and multiplication. $\mathbf{R} \times \mathbf{R} \to \mathbf{R}$ As inputs, they accept two real values and output a third real number. The operation of division is very close to being a binary one; but, given that division by 0 is not specified, we can only refer to it as a partly defined binary operation. The vast majority of our activities, such as division, will be specified in every location, but a few of them, like, won't be.

The negation of a statement may demonstrate unary operations, often known as functions. $\mathbf{R} \to \mathbf{R}$ A single real number is inputted and a single real number is outputted by it. The operation of reciprocation is regarded as a partial unary one as the zeroth reciprocal does not have a definitive value.

All of the operations that we are going to look at are either binary or unary. There is no question that ternary operations may be specified, but in practice, ternary operations are seldom helpful.

Famous people's identities are brought to light in some of these procedures. For example, due to their same identity, the addition and multiplication operations are commutative.

$$x + y = y + x \quad \text{and} \quad xy = yx.$$

It is claimed that a binary operation is commutative when it does not make a difference which order the two inputs are applied in; in other words, when switching them, also known as commuting one across the other, does not affect the outcome in any way. However, division and subtraction do not behave in a commutative manner.

$$(x + y) + z = x + (y + z) \quad \text{and} \quad (xy)z = x(yz).$$

An associative binary operation is one that accepts three parameters and always produces the same result. The word "associative" is used to designate this kind of operation. As a consequence of this, the brackets may be connected to either the first or the second group of binary variables. The use of associativity is not permitted in operations involving subtraction or division.

Additionally, identity is a component of both addition and multiplication by definition.

$$0 + x = x = x + 0 \quad \text{and} \quad 1x = x = x1.$$

In the context of binary operations, an element is referred to as a "identity element" if, when it is included in the operation, it does not alter the values of any other elements that are included inside the set. In addition, this kind of element is frequently referred to as a "neutral element." When the identity element is being added or multiplied, it is equal to zero, and when it is being divided, it is the identity element itself. It is important to note that the division and subtraction methods do not include any aspects of identity. On the other hand, at the very least, that is because $x - 0 = x$ and $\frac{x}{1} = x$, but not on the left, since usually $0 - x \neq x$ and $\frac{1}{x} \neq x$.) In addition, there is an inverse addition and an inverse multiplication for each and every item that is not zero. To put it another way, there is an extra component, denoted by the symbol - x, that guarantees the truth of any given value of x. $x + (-x) = 0$, for all non-zero values of x, additional element is present, namely $\frac{1}{x}$ such that $x\frac{1}{x} = 1$. For this reason, we may state that an operation is complete when the identity element is generated by combining the inverse elements with the original components using the operation being described, provided that the operation comprises inverses for each element in the operation. By doing addition and multiplication in a different manner, it is feasible to get the inverses of operations that include things that are not zero.

In conclusion, the mathematical operations of addition and multiplication are differentiated from one another by a certain connection that is referred to as distributivity.

$$x(y+z) = xy + xz \quad \text{and} \quad (y+z)x = yx + zx.$$

It is possible to divide the value of x among the components of a total by multiplying it by x. This is something that may be done. The fact that addition is more difficult than multiplication is the reason why this is the case.
My first workout is: We will discuss what operations are in this section.

a. Does it happen using a binary process $x * y = \frac{xy}{x+y}$ Can a commutative operation be constructed for the positive integers x and y? Is this something that can be done? With that being stated, do you believe that it is correct to say that $x * y = y * x$ if both x and y are positive, what happens? What kind of connection do they share? Please elaborate on your response to my question.

b. Is it true that $(w - x) - (y - z) = (w - y) - (x - z)$ is there a formula for identifying real numbers? Are you able to present a cause for you to support or oppose something? (When we state that an equation has "identity" when both sides are defined and equal, we imply that the equation holds true in every circumstance.)

c. Although the distribution of multiplication is distributed over the distribution of addition in R, the distribution of addition is smaller than that of multiplication. Offer an illustration of the reasons why it would not be appropriate.

Fields in a Nutshell

When we talk about "division," we mean that we may split the result by any nonzero field element. In particular, the same rules of commutativity, associativity, and distributivity as one is acquainted with from college algebra lectures are necessary to be satisfied by these operations in order for them to be considered "nice." A few pages later, we shall provide a series of axioms that explain in further detail what is meant when we speak of a "field." In terms of logic, it makes perfect sense to begin with the exact definition; but, in terms of pedagogy, I believe that it is preferable to offer instances first so that the learner can understand what it is that is being abstracted before delivering the abstraction itself. The following are some prominent instances of fields: R, which is the field of real numbers ;

- the field Q of rational numbers; and
- the field C of complex numbers. Note that Q ⊂ R ⊂ C. Also we have
- finite fields, also known as Fq, which are fields that contain a limited number of objects $q = p^r$ where p is prime and $r \geqslant 1.$

In addition to these, there are a much larger number of other disciplines. It is important to keep in mind that the ring of integers, which is represented by the letter Z, is not a field since the operation of division cannot be used to close it. There is no such thing as an empty ring, yet every field is one. First, it will be helpful to explore the particular instances of fields described above, searching for common aspects as well as qualities that are unique to each example of a field. We will define more specifically what it is intended to imply by a 'field,' or a 'ring,' and then continue to prove general statements about fields.

Just to be clear, for each positive integer n, the set will always include at least one member with the above properties if F is any field.

$$F^n = \{(a_1, a_2, \ldots, a_n) : a_1, a_2, \ldots, a_n \in F\}$$

With the help of Gaussian elimination, for instance, we might solve systems of linear equations that have coefficients in F, as shown in Math 2250 (Elementary Linear Algebra), which is the course that we are now taking. Z, which often contains an unknown a/b, is not an exception; this method does not work on generic rings either. Therefore, one of the reasons we choose using fields rather than rings is so that we can solve linear problems.

OBJECTIVES

1. To study mathematical formulation
2. To study multiple field concepts in contemporary algebra

The Actual Quantities

The real number system, denoted by the letter R, is comprised of all of the integers, all of the rational numbers, and a great deal of additional numbers. For example,

$$3,\ -11,\ 0,\ \tfrac{31}{6},\ -7.1556,\ \sqrt{2},\ \pi,\ \frac{17-\sin(14.2)}{1+\sqrt{6}}\ \in \mathbb{R}.$$

It does not include the complex number $i = \sqrt{-1}$; and it does not include ∞ or any infinitesimal values.

What precisely is meant by the term'real number'? One of the objectives of this class will be to provide an adequate definition of R; nevertheless, let's not attempt to solve this problem just now. To begin, let's just go over some of the most fundamental aspects of R, including what sets it apart from many other areas. The field R has complete ordering.

This means that given any two real numbers a, b ∈ R, a single relation out of three possible ones is true:

$$a < b, \quad a = b, \quad \text{or} \quad a > b.$$

(We call this the trichotomy property.) The relation '< b and c > 0, then ca < cb. Most fields do not share this property; in particular, C and Fq are not ordered. (We cannot have i > 0 since this would force −1 = i 2 < 0, a contradiction; and if −i < 0 then we may obtain a similar contradiction.) The field Q is ordered, but Q differs from R in other important ways as we'll soon observe.

There is no further work to be done in the subject of R. We won't go into too much detail on this key characteristic of the real number system just yet, but we will mention that the Intermediate Value Theorem of Calculus is based on one of the qualities of the real number system. This is something that we will point out. Given that f(1) is less than zero and f(2) is greater than zero, there is a real number c that falls between 0 and 1 that is such that f(c) equals zero. This assertion is no longer accurate if we substitute the word "real" with the word "rational." The most crucial thing to note is that R is complete, while Q is not fully formed. Details will be provided later.

It is not possible to count the value placed in the R field. The fact that we are unable to disclose the real number is an indication of this. In the case of R = {a1, a2, a3,...}, it is simply not possible to list all of the real numbers in this manner. Because the rational numbers may be named in an endless series, this feature also differentiates R from Q. In other words, Q is countable, but R must be countable. In a moment, we will also comment on this particular subject.

However, R is not the only field that has the properties of being uncountable, totally ordered, and complete; other fields also possess features that are comparable to these. For the time being, the issue is: how do we define R? Despite the fact that one could be inclined to imply that real numbers correspond to points on a physical line, this definition is not adequate for a number of different reasons. It is not possible to find points on a physical line that match to real numbers. The first thing to note is that this intuitive explanation is so nebulous and unclear that it is not even remotely capable of being used in a demonstration of anything. Furthermore, this assertion is not true even in the most literal and physical definition of the word. Despite the fact that it may be difficult for you to acknowledge this fact, points on a physical line in space or in time are not entirely arranged!

Another prominent approach to defining real numbers makes use of decimal representations by assigning a value to each real number based on the expansion it receives in decimal form. This is an improvement, however it is not adequate for a number of reasons, including the following:

It is possible that real numbers include more than one expression into their decimal expansion; being aware of this fact might assist you in finding the appropriate answer. To better understand my point of view, have a look at these five sentences.

$$1, \qquad 01, \qquad 1.00, \qquad 1.000000\ldots, \qquad \text{and} \qquad 0.999999\ldots$$

all are equivalent representations of the same actual integer. (The equivalence 0.9999... = 1 is in itself a significant truth that we are going to have to talk about very soon! One is incapable of comprehending the subject matter of the real number system before coming to grips with this reality.

Even if the preceding technical challenge of non-unique decimal representations can be overcome, one is still faced with the unsettling reality that, despite the usefulness of decimal expansions for day-to-day applications, they are of virtually no help in establishing even the most fundamental truths about R. This is true even if the previous technological challenge can be overcome. Decimal expansions are of little use, for instance, if one wishes to demonstrate that the Intermediate Value Theorem is true. Therefore, you should make sure that your definition of R does not include any references to decimal expansions. Consider the fact that even if f is continuous, which means that moderate shifts in x result in moderate shifts in the output, in f(x), this fact is not evident in the decimal digits, where relationship between the decimal digits of x and f(x) can be very mysterious and hard to predict. (Consider that the decimal digits of x are not related to the decimal digits of $f(x) = x^2 - 2$ in any obvious way. there is no clear description of the decimal digits of $x = \sqrt{2}$. Nobody even knows if infinitely many decimal digits of $\sqrt{2}$ are equal to 3!)

Theorem (d.b. Coleman, 1966)

Here we may assume that G is an n-order finite group and that R is an integral domain with 1. Then we can continue. The group algebra RG cannot be broken down into its component elements unless and until every prime divisor of n is a non-unit in R. We will refer to this group as G. Define the order in which the subgroups are arranged by using the notation.

(a) The commutator subgroup of G is G(l).

(b) EG(i) = subgroup of G produced by aba-1b-1, in which EG(i-I) and EG(b) are

G is considered nilpotent if and only if G(k) = (e) for every k > 1.
If the subgroup G/A is abelian for any normal subgroup A of G that is abelian, then G is called metabelian.

If group G is finite, then every locally finite subgroup of G that is created in a finite way is also locally finite.

Theorem (d. A jackson, 1969)

Consider the possibility that G is a finitely nilpotent group or a finite metabelian. With that out of the way, we may give the following qualities the same weight.

Property A. Consider the group G to be either a finite metabelian or a f mite nilpotent one. If this is the case, the following characteristics are equivalent

Property B. Every locally finite group of ZG units has an isomorphic counterpart in ZG in the form of a group of trivial unit groups.

PROPERTY CIn the event that the following requirements are satisfied and u G is a unit of the finite order of ZG:

$$u_G \equiv 1 \text{ mod} A(G)A(G,G'),$$

Theorem (sudarshan k. Seghal,1969)

It must be remembered that the only finite-order units in the integral group ring z G of any abelian group G are of the form ±t, where t is a torsion element.

We prove in the next paragraphs that the idempotents of the rational group ring QG and the structure of the unit group of the integral group ring z G are the same.

Theorem (Raymond Ayoub And Christine Ayuob,1969)

Several factors should be taken into account: "G" stands for an n-order fmite abelian group. The definition of U(ZG) is equivalent to the product of ±G and F for a free abelian group F of rank 12(n+1+tr 2s), where t2 is the number of elements of order 2 and s is the number of cyclic subgroups of G. This is due to the fact that t2 stands for the quantity of components of order 2 linked to G.

The second step is to assume that any finite abelian group G has index m. When there exist td cyclic subgroups of rank d and Qd cyclotomic fields, and when m has a divisor d, then the following will be true:

$$QG \cong \bigoplus_{d|m} t_d Q_d .$$

''' d dim

(c). For any finite abelian group G such that G/D is cyclic, consider the subgroup D. $u_D \in QG$ represent the identification of the idempotent imitate $\pi_D(QG)$, the projection of QG onto Qd, where the corresponding character has kernel D, and for $K \subseteq D$ l et.

$$e_K = \sum_{K \subseteq D,\ G/D \text{ cyclic}} u_D$$

There

$$e_K = (1/|K|) \sum_{k \in K} k$$

$$u_D = \sum_{D \subseteq H} \mu(|H/D|) e_H$$

where μ is the Mobius μ - function.

History of problem and statements of results

The discriminant is a useful tool for finding a Z-basis of the algebraic integer ring AK in the setting of an algebraic number field K. A Z-basis of AK, simultaneously called an integral basis of K in certain contexts, might be called discriminant in others. There may be occasions when doing this activity presents challenges. For a long time, its computation has captivated a large number of mathematicians, particularly those interested in pure algebraic number fields. A possibility of pure cubic fields with an integral basis was shown by Dedekind [Ded2] in 1900.

Westlund Wes successfully laid the integral groundwork for a discipline of pure prime degree mathematics in 1910. Funakura Fun found a possible formula in 1984 that might be used to find an integral basis for all pure quartic fields. They laid the groundwork for their future work in 2015 when Hameed and Nakahara [Ha-Na] solidified the basis for these pure octic fields Q(√8 a). This laid the groundwork for what was to come in their careers. For the sake of this argument, a free square integer is defined as. The goal here is to find primes p such that n is divisible by either an or vp(a), where vp(a) is the largest power of p that divides a, and p is coprime. In order to do this, a scenario will be developed.

For the purpose of obtaining the discriminant of n-th degree fields of type Q(−n a), we have the possible option of using this knowledge to formulate a formula. It is important to take into consideration the fact that it is said unequivocally that these areas possess an intrinsic base. Taking into consideration this is still another problem that has to be done. In order to develop the formula for an integral basis of n-th degree fields of the type Q(√n a) with a, n coprime, K has established the formula. Okutsu in a series of publications. However, we provide counter cases to show that this formula is erroneous (for examples, see instances 4.3.9 and 4.3.10). With the exception of a few preliminary discoveries, the discriminant formula for K = Q(−n a). we provide a demonstration of the theorem that elucidates an integral basis of K, as well as a few applications of this theory. At the end of the chapter, we show our method to computing based on integral foundation by providing several examples.

CONCLUSION

In order to illustrate a wide range of topics in contemporary algebra, this thesis made use of number theoretic systems. The mathematical curriculum that is now being taught in junior and senior high schools covers the number theoretic systems that are being utilized in the examples. On the other hand, the reader who is not acquainted with these systems will not need a significant amount of time to acquire sufficient knowledge to correctly understand the examples. By providing the reader with these pictures, the author hopes to improve their understanding of modern mathematics as well as their appreciation for it.

Some consideration was given to it. The orifical frameworks such as fields, integral domains, rings, groups, and ideals The first chapter of the dissertation discussed the author's method, the content of the dissertation, and the significance of the dissertation. The purpose of Chapter II was to investigate groups, subgroups, as well as isomorphisms and homomorphisms that exist inside groups. The topics that were discussed in Chapter III were rings, subrings, ring homomorphisms and isomorphisms, ideals, and certain types of rings. The fourth chapter of the book included topics such as integral domains, subdomains, and fields. Throughout the whole of the dissertation, references were made to definitions, theorems, and examples that have previously been presented. This was done in order to avoid repetition and to draw links between the many topics that were discussed.

REFERENCES

1. Bertram, E. A. (1983). Some applications of graph theory to finite groups. In *Discrete Mathematics*, *44*(1), 31–43. https://doi.org/10.1016/0012-365X(83)90004-3
2. Bondy, J. A., & Murty, U. S. R. (1976). *Graph theory with applications*, *290*. Macmillan. https://doi.org/10.1007/978-1-349-03521-2
3. Brouwer, A. E., & Haemers, W. H. (2011). *Spectra of graphs*. Springer Science+Business Media.

4. Cardoso, D. M., de Freitas, M. A. A., Martins, E. A., & Robbiano, M. (2013). Spectra of graphs obtained by a generalization of the join graph operation. In *Discrete Mathematics*, *313*(5), 733–741. https://doi.org/10.1016/j.disc.2012.10.016
5. Cayley, P. (1889). On the theory of groups. In *American Journal of Mathematics*, *11*(2), 139–157. https://doi.org/10.2307/2369415
6. Chakrabarty, I., Ghosh, S., & Sen, M. K. (2009). Undirected power graphs of semigroups. In *Semigroup Forum*. Springer, *78*(3), 410–426. https://doi.org/10.1007/s00233-008-9132-y
7. Chang, K.C., Shao, S.H., & Zhang, D. (2017). Cheeger's cut, maxcut and the spectral theory of 1-Laplacian on graphs. In *Science China Mathematics*, *60*(11), 1963–1980. https://doi.org/10.1007/s11425-017-9096-6
8. Chartrand, G., Eroh, L., Johnson, M. A., & Oellermann, O. R. (2000). Resolvability in graphs and the metric dimension of a graph. In *Discrete Applied Mathematics*, *105*(1–3), 99–113. https://doi.org/10.1016/S0166-218X(00)00198-0
9. Chattopadhyay, S., & Panigrahi, P. (2015). On Laplacian spectrum of power graphs of finite cyclic and dihedral groups. In *Linear and Multilinear Algebra*, *63*(7), 1345–1355. https://doi.org/10.1080/03081087.2014.936435
10. Chattopadhyay, S., Panigrahi, P., & Atik, F. (2018). Spectral radius of power graphs on certain finite groups. In *Indagationes Mathematicae*, *29*(2), 730–737. https://doi.org/10.1016/j.indag.2017.12.002
11. Chattopadhyay, S., Patra, K. L., & Sahoo, B. K. (2020). Laplacian eigenvalues of the zero divisor graph of the ring Zn. In *Linear Algebra and Its Applications*, *584*, 267–286. https://doi.org/10.1016/j.laa.2019.08.015
12. Chvátal, V. (1973). Tough graphs and Hamiltonian circuits. In *Discrete Mathematics*, *5*(3), 215–228. https://doi.org/10.1016/0012-365X(73)90138-6
13. Cui, S.-Y., Tian, G.-X., & Guo, J.-J. (2013). A sharp upper bound on the signless Laplacian spectral radius of graphs. In *Linear Algebra and Its Applications*, *439*(8), 2442–2447. https://doi.org/10.1016/j.laa.2013.06.015
14. Cvetković, D., Rowlinson, P., & Simić, S. (2009). *An introduction to the theory of graph spectra*. Cambridge University Press. https://doi.org/10.1017/CBO9780511801518
15. Sachin, S. (2024). SUSTAINABLE UNINTERRUPTED LEARNING – AN APPROACH TO BLENDED LEARNING. *Shodh Sari-An International Multidisciplinary Journal*, *03*(02), 86–101. https://doi.org/10.59231/sari7690
16. Cvetkovíc, D., Rowlinson, P., & Simíc, S. K. (2007). Signless Laplacians of finite graphs. In *Linear Algebra and Its Applications*, *423*(1), 155–171. https://doi.org/10.1016/j.laa.2007.01.009

17. Akujieze, M. O. (2024). Innovative teaching techniques for alleviating mathematics phobia among secondary school students in Anambra State. *Edumania-An International Multidisciplinary Journal*, *02*(03), 116–132. https://doi.org/10.59231/edumania/9060
18. Cvetkovic, D. M. et al. (1988). *Recent results in the theory of graph spectra*. Elsevier.
19. Bhagoji, M. D. (2024). Navigating Global Dynamics in Teacher Education: A Comprehensive Overview. *Shodh Sari-An International Multidisciplinary Journal*, *03*(01), 123–133. https://doi.org/10.59231/sari7660
20. M., A. P., & C, K. D. (2024). Applied mathematics in today's education system in India. *Shodh Sari-An International Multidisciplinary Journal*, *03*(01), 113–122. https://doi.org/10.59231/SARI7656