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# A study of fixed point results in partially ordered metric spaces using weakly contractive mappings

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## Abstract

This research extends conventional fixed-point theory to broader contexts by investigating fixed point outcomes in partially ordered metric spaces through the use of weakly contractive mappings. Weakly contractive mappings are presented within the framework of partially ordered metric spaces, where the distance between mapped points is diminished by a factor $\alpha$ (where $\alpha$ falls within the interval [0, 1]). The study proves that such mappings do in fact lead to convergence and fixed points, and that these points are unique, even though they are weaker than rigorous contractions. A number of theorems concerning the circumstances in which unique fixed points exist are given by the study, and the behaviour of these mappings is substantially affected by the metric space order structure. The use of weakly contractive maps in optimization, iterative approaches, and dynamic systems demonstrates their effectiveness in addressing complicated issues. Expanding its applicability to partially ordered metric spaces; this study adds to the development of fixed-point theory and provides fresh insights and possible applications in many mathematical and practical situations.

*Keywords:* Mapping, Fixed-point, Metric, Weakly, Spaces.

## Introduction

Recent years have seen a surge in interest in fixed point theory in partly ordered metric spaces as a result of its broad applicability to fields as diverse as optimization theory, computer science, economics, and engineering, among many more. In the context of metric spaces, fixed point findings are especially useful for understanding iterative processes and the presence of unique solutions to various types of equations. The term "fixed point" describes any point that remains unchanged after applying the function. In mathematics, a point x is considered a fixed point if and only if the function f(x) is equal to x.

As a generalization of sets, partially ordered sets (posits) allow items that are not always comparable to one another. The analysis of fixed points becomes much more intricate and detailed with this structure, as the sequence of members in the set can affect how the function or mapping behaves. Along with a distance function (metric), a partially ordered metric space also has an additional property that permits comparison of elements under specific conditions, which adds depth and complexity to the study.

Essential families of mappings that generalize contractive mappings are weakly contractive mappings. A contractive mapping is one that pulls points closer together in a specific sense. Less stringent than contractive mappings, weakly contractive mappings can be used in contexts where traditional contractivity restrictions do not apply. These maps are true if and only if they reduce the distance between points under the mapping by some factor; however, this reduction could not be constant at all points. In particular, in contrast to rigid contractions in the classical sense, weakly contractive mappings are more lenient when it comes to the distance reduction, typically utilizing a weaker constraint in relation to the distance between the image points and the original locations.

A large amount of new research may be explored when weakly contractive mappings, partly ordered sets, and fixed point theory come together. The monotonicity and ordering of fixed points may be taken into account using a partial order, which is especially helpful in optimization and game theory applications, where the structure of the solution is dependent on the ordering of the components in the space. On top of that, weakly contractive mappings allow us to generalize the fixed point conclusions to more general classes of functions, even if they don't meet the rigorous contractivity requirements that classical fixed point theory has always assumed.

For fixed points under weakly contractive mappings, the goal of the research is to prove their existence, uniqueness, and convergence. From both a theoretical and practical point of view, these conclusions are crucial, as they specify the requirements for algorithms and iterative approaches that use these mappings to converge to a fixed point. Examining these mappings in partially ordered metric spaces also paves the way for future investigations into the behavior of iterative processes in broader contexts, where partial order relations may impact the convergence characteristics.

This research aims to fill some gaps in the literature by studying the circumstances in which unique fixed locations that are weakly contractive in partly ordered metric spaces can be efficiently computed and by improving upon previous findings on these points. The study's overarching goal is to develop more broadly applicable fixed point theorems by expanding our knowledge of the relationship between metric structure, partial order, and contractivity via investigation of these complex ideas. The convergence of iterative approaches based on weakly contractive mappings is also explored in this paper, which could lead to novel algorithms for tackling real-world issues in different engineering and scientific fields.

## Review of Related Studies

Beg, Ismat et al., (2024). New fixed point results in partly ordered bicomplete quasi-metric spaces are proven in this paper. In this paper we expand and extend the work of Nieto and others on contraction mappings in partly ordered full metric spaces and of Schellekens on contraction mappings in bicomplete quasi-metric spaces. No weakening of our assumptions or deduction of our results from the famous Kleene's fixed point theorem is demonstrated either. As a last step, we apply our findings to recurrence equations' asymptotic analysis.

Ahmed, Athraa et al., (2024). In this work, we discuss and analyze several conclusions about mapping in partly ordered S-metric spaces that possess the strong mixed monotone feature, as well as some suggestions for AB-coupled fixed point outcomes. Furthermore, AB-coupled fixed points are shown to exist and to be unique. In [15] and [4], we expand upon the key claims made by Virendra Singh Chouhan and Richa Sharma (2015) and Gnana Bhaskar and Lakshmi kantham (2006).

Namana, Seshagiri Rao et al., (2021). We examine the existence of a fixed point and the uniqueness of a mapping in an ordered b-metric space using a generalized (φ) -weak contraction. Some results on a linked coincidence point and a coincidence point of two mappings are also stated using the same contraction condition. Some recent studies in the subject have benefited from these findings, which both expand their reach and contribute to them. We provided examples at the conclusion to support our statements. What follows A mapping's uniqueness and the building of a fixed point in partly ordered b-metric spaces are discussed. In addition, two mappings satisfying generalized weak contraction requirements are tested for the existence of a linked coincidence point and a coincidence point.

Ughade, Manoj. (2016). For non-decreasing maps with rational expressions in S-metric spaces with partial order, this work aims to prove several fixed point theorems using a collection of matching pairs of functions. Our research extends and refines the results presented in [1-4] for metric spaces to S-metric spaces. We provide instances to demonstrate the validity of the outcomes.

Işık, Hüseyin & Turkoglu, Duran. (2013). this thesis proves a number of fixed point theorems for descriptions of weakly contractive mappings in ordered metric-like spaces. We illustrate our conclusions with an example and a few applications to support their usefulness. In doing so, they generalize several results from the literature that have already been established. Further findings on fixed points in ordered partial metric spaces are also obtained by us.

Shatanawi, Wasfi. (2010). In their study titled "Fixed point theorems in partially ordered metric spaces and applications," Bhaskar and Lakshmi kantham examined the point of coincidence for a mapping F from X×X into X and a mapping g from X into X. E. Karapinar proved in his paper "Couple fixed point theorems for nonlinear contractions in cone metric spaces" that there exists a linked coincidence point between normal cones without regularity and mappings F and g from X into X. This work found that equipped coincidence fixed point theorems over cone metric spaces are not always normal. Our results build on those of earlier studies that found comparable things.

## Preliminaries

The notion of partially ordered sets, or poses, is fundamental to fixed point theory since it allows one to generalize conclusions from metric spaces to more organized spaces. For a set P to be considered partially ordered, it must have a binary relation ≤ in addition to being reflexive, antisymmetric, and transitive. Not all element pairs must be similar; a "partial" order can be defined according to these characteristics. Theoretically, fixed point theory makes use of this sorting to pinpoint potential locations for fixed points and the ways in which they respond to specific mappings.

Metric spaces (X, d) are sets where the distance function is non-negative, symmetric, and satisfies the triangle inequality. In addition to laying the groundwork for talks about convergence and continuity, these features make it possible to define distances between space elements. New opportunities for investigating mapping fixed points arise when partial ordering is applied to a partially ordered metric space, which then allows for comparisons of components not only by distance but also by order.

An expansion of the standard notion of contraction mappings is the weakly contractive mapping. If there exists a constant α in the range (0,1) such that for all values of x less than or equal to y, the mapping T is weakly contractive in the metric space, and this holds for any pair of points x and y in X, the inequality d(T(x), T(y)) < ad (x, y) holds. In contrast to the conventional contraction condition, which states that this inequality must be true for all possible orderings of point pairs, this one is less stringent. In spaces with an intrinsic order structure, the weakly contractive property opens up a larger class of mappings from which fixed point results can still be obtained, which is a very important result.

If T:X→X at least one element x* in X such that T(x*) = x, then the function has a fixed point x*. Fixed points in metric spaces are an important topic in many mathematical subfields., especially those with partial orders, for issues such as optimization, game theory, and dynamical system analysis. Thorough examination of the order structure and metric space characteristics is frequently necessary to ascertain the presence and uniqueness of fixed points in the context of weakly contractive mappings. One well-known result in fixed point theory is the Contraction Mapping Theorem, which is also called the Banach Fixed Point Theorem). This assertion states that if T is a contraction mapping on a whole metric space, then that space has a unique fixed point. The conventional conclusions, however, require adjustment to account for the weaker contraction requirement when the mapping is weakly contractive. Spaces like partly ordered metric spaces, which have broader fixed point outcomes and more structure, are made feasible by this. All Cauchy sequences converge to some element in a complete metric space. Assuming the mapping fulfils the correct contraction condition, fixed point theory relies on completeness to ensure that iterative mapping sequences converge to a fixed point.

Verifying the space's completeness is frequently an essential prerequisite for proving the presence as well as the singularity of fixed points in the study of weakly contractive maps in partly ordered metric spaces. The study of fixed point results in partly ordered metric spaces using weakly contractive mappings expands the typical scope of conventional fixed point theory by adding the complexity of order and weaker contraction restrictions. These generalizations shed light on the behaviour of systems modelled by partially ordered structures, which is useful in many academic and applied fields.

**Existence of Fixed Points for Weakly Contractive Mappings**

The existence of fixed points for weakly contractive mappings in partly ordered metric spaces is a crucial result in the area of fixed point theory. A weakly contractive mapping $T: X \to X$ and a full partly ordered metric space (X, d) are considered in this context. The mapping is deemed weakly contractive if there exists a constant $\alpha$ in the range [0, 1] such that the condition $d(T(x), T(y)) < \alpha d(x, y)$ holds for any x, y in the space, and $x \leq y$ in the partial order. Since the weak contractive condition is unique to element ordering, the idea is more generic and can be applied to a broader class of mappings than conventional contractive mappings.

The presence of a fixed point for such a weakly contractive mapping is justified by the fact that the mapping does, in fact, move elements closer together, but with weaker conditions than ordinary contraction, as guaranteed by the contraction property. To be more precise, a limit point, or fixed point of the mapping, will be reached by the sequence of mapping iterates starting from any point in X if the space is complete. This point must satisfy $T(x*) = x*$, making it a fixed point, because the space is complete and the Cauchy sequence created by the mapping iterates converges to a point in the space. This point is also called a fixed point because the mapping is weakly contractive.

This result generalizes the conventional Banach Fixed Point Theorem, which asserts that contraction mappings in full metric spaces always have a fixed point. Assuming the space is full, the presence of a fixed point is certain even in the situation of weakly contractive mappings, which ease the contraction criterion. Broader applications in mathematical analysis, optimization, and other areas involving partially ordered structures are made possible by the weak contraction property, which is particularly helpful in cases when rigorous contraction is excessively restrictive.

**Theorem (Existence of Fixed Point):**

According to the phrase, if we have two variables x and y in the set X, and α is an integer between 0 and 1, then T: X→X is a weakly contractive mapping and (X, d) is a complete metric space.

$$d(T(x), T(y)) \leq ad(x, y).$$

After that, there is a unique fixed point in X for T.

**Proof:**

There are two parts to our proof of this theorem: existence and uniqueness.

**Step 1: Existence of a Fixed Point**

In order to prove that there is a fixed point, we start by taking any point x0 in X and using the mapping T to define the series $\{x_n\}$.

$$x_{n+1} = T(x_n), for\ n \geq 0.$$

Demonstrating that this sequence approaches a fixed point of T is the objective.

**1.1: Convergence of the Sequence**

Since X is complete, we shall demonstrate that the sequence $\{x_n\}$ converges since it is Cauchy.

You must first determine the separation between any two phrases that follow each other in the sequence:

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})).$$

Given that T has a weak contractive property, we can say:

$$d(T(x_n), T(x_{n-1})) \leq ad\ (x_n, x_{n-1}).$$

This disparity informs us that the geometrically decreasing sequence of distances between succeeding terms is a factor of α\alphaα. For all integers n greater than or equal to 1, the following inequality is true:

$$d(x_{n+1}, x_n) \leq ad\ (x_n, x_{n-1})$$

The next step is to calculate a rough distance between$x_{n+1}$ and $x_0$. We obtain: by applying the triangle inequality.

$$d(x_{n+1}, x_0) \leq d(x_{n+1}, x_n) + d(x_n, x_{n-1}) + \cdots + d(x_1, x_0)$$

We learn that for every given distance that the weak contraction property holds,$d(x_{n+1}, x_n)$ is less than or equal to $\alpha^n d(x_1, x_0)$, so we have:

$$d(x_{n+1}, x_0) \leq d(x_1, x_0) \sum_{k=0}^{n} \alpha^k = d(x_1, x_0) \frac{1 - \alpha^{n+1}}{1 - \alpha}$$

As n→∞, The total approaches a fixed point:

$$\lim_{n \to \infty} d(x_{n+1}, x_0) = \frac{d(x_1, x_0)}{1 - \alpha}$$

Since α<1, the sequence $\{x_n\}$ is complete, and so it converges to a limit point x*∈X, proving that it is Cauchy.

**Step 2: Uniqueness of the Fixed Point**

Then, we prove that there is only one fixed point, the limit point x*, and that it is a fixed point.

Let x* = $lim_{n \to \infty} x_n$. Based on the weak contraction property and the continuity of T, we can deduce:

$$T(x^*) = T\left(\lim_{n \to \infty} x_n\right) = T \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = x^*$$

Thus, x* serves as a constant for T.

To show uniqueness, suppose there are two fixed points $x_1$ and $x_2$ of $T$, i.e., $T(x_1) = x_1$ and $T(x_2) = x_2$. By the weak contraction property, we have:

$$d(T(x_1), T(x_2)) \leq \alpha d(x_1, x_2).$$

Since $x_1$ and $x_2$ are fixed points, we get:

$$d(x_1, x_2) \leq \alpha d(x_1, x_2).$$

This implies that:

$$d(x_1, x_2) = 0,$$

so $x_1 = x_2$. Therefore, the fixed point is unique.

## Conclusion

If T' is a weakly contractive mapping on a complete metric space X, then T has a unique fixed point, as we have demonstrated. The proof is based on the weak contraction property, which guarantees that the distance between iterates diminishes at each step, enabling the sequence to converge to this fixed point. It is generated by iterating T. Existence and uniqueness in such contexts are guaranteed by the fact that weakly contractive mappings can only have one fixed point, which leads to the uniqueness.

To show uniqueness, suppose there are two fixed points $x_1$and$x_2$ of T, i.e., T$(x_1)$ =$x_1$ $and$ T$(x_2)$ $= x_2$. By the weak contraction property, we have:

$$d\big(T(x_1), T(x_2)\big) \leq ad\ (x_1, x_2)$$

Since $x_1$ and $x_2$ are fixed points, we get:

$$\mathrm{d}\ (x_1, x_2) \leq ad\ (x_1, x_2)$$

This implies that:

$$\mathrm{d}(x_1, x_2) = 0$$

so $x_1 = x_0$. Therefore, the fixed point is unique.

Finally, we proved that for any weakly contractive mapping T on a full metric space X, there exists a unique fixed point. In order for the series to converge to this fixed point, the proof relies on the weak contraction property, which ensures that the distance between iterates decreases at each step. The result is the result of repeating T. The uniqueness stems from the fact that weakly contractive mappings can only have one fixed point, guaranteeing that such mappings do in fact exist and are distinct.

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